Modélisation et Analyse de Nouvelles Extensions pour le Problème du Vendeur de Journaux

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Modeling and Analysis of New Extensions for the News-Vendor Problem

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Titre : Modélisation et Analyse de Nouvelles Extensions pour le Problème du Vendeur de Journaux

Mots clés : Gestion de stock, Problème du Vendeur de Journaux, Multiples soldes, Substitution, Assortiment, Drop-shipping

Le NVP (Problème du Vendeur de Journaux) a été étudiée de façon continue au cours des dernières décennies pour la prise de décision dans les industries manufacturières et de services. Bien que beaucoup de travail a été fait dans le domaine du NVP, l'intérêt sur ce sujet ne diminue pas. Alors que de nouvelles tendances émergent dans les affaires, par exemple flux internationaux de produits et de e-commerce, les détaillants sont confrontés à de nouvelles situations et la littérature de NVP doit être enrichi.
Dans ce travail, nous proposons trois nouvelles extensions NVP compte tenu des questions importantes rencontrées par le NV: plusieurs soldes, variété de produits et d'assortiment ainsi que des problèmes de drop-shipping et de retour des produits qui sont liés à l'e-commerce. Notre travail ajoute de la valeur à partir des travails antérieurs dans plusieurs aspects: assouplissement des hypothèses, l'examen de nouvelles questions, de nouvelles formulations et de la méthodologie ainsi que des aperçus intéressants. Nous formulons les modèles et donner les conditions d'optimalité de la quantité de commande. Aperçus utiles sont fournis sur la base des études numériques.

Title : Modeling and Analysis of New Extensions for the News-Vendor Problem

Keywords : Inventory management, News-Vendor Problem, Multiple discounts, Substitution, Assortment, Drop-shipping

The NVP (News-Vendor Problem) has been continuously studied over the last decades for decision making in manufacturing and service industries. Although a lot of work has been done in the NVP area, interest on this topic does not decrease. As new trends emerge in business, e.g. international flow of products and e-commerce, retailers are facing new situations and the literature of NVP needs to be enriched.
In this work, we propose three NVP extensions considering important issues faced by the NV: multiple discounts, product variety and assortment as well as drop-shipping and product return problems that are related to e-commerce. Our work adds value from earlier achievements in several aspects: relaxation of assumptions, consideration of new issues, new formulations and methodology as well as interesting insights. We formulate the models and give the optimality conditions of the order quantity. Useful insights are provided based on numerical studies.

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Abstract

The News-Vendor Problem (NVP) has been continuously studied over the last decades for the decision making in manufacturing and service industries. Although a lot of work has been done in the NVP area, interest on this topic does not decrease. As new trends emerge in business, e.g. international flow of products and e-commerce, retailers are facing new situations and the literature of NVP needs to be enriched. In this work, we propose three new NVP extensions considering important issues faced by the NV: multiple discounts, product variety and assortment as well as drop-shipping and product returns problems that are related to e-commerce. Our work adds value from earlier achievements in several aspects: relaxation of assumptions, consideration of new issues, new formulations and methodology as well as interesting insights. We formulate the models and give the optimality conditions of the order quantity. Useful insights are provided based on numerical studies.

In particular, for dealing with overstock, we present a NVP model with price-dependent demand and multiple discounts. We prove the concavity of the expected profit on order quantity under general demand distributions. The optimal initial price and discount scheme are also analyzed. The product variety is treated in a multi-product NVP with demand transfer (the demands of products not included in the assortment proposed in the store are partly transferred to products retained in the assortment) and demand substitution between products that are included in the assortment, by focusing on the joint determination of optimal product assortment decision and optimal order quantities for products that are included in the assortment to optimize the expected total profit. For e-commerce, we consider a NV managing both a physical store inventory and a sale channel on internet that is fulfilled by a drop-shipping option, as well as the possibility of reselling products that are returned by consumers during the selling season. The concavity of the expected profit is proven and various results are obtained from a numerical analysis.

Some managerial insights are derived from these models: using multiple discounts can increase the expected profit remarkably and it is shown that it is better to decrease the selling price slowly in the beginning of the selling season. The increase of the fixed cost related to including a product variant in the assortment will reduce the optimal assortment size and also the expected profit. Moreover, drop-shipping, can bring some important increase of the expected profit.

Key words: *Inventory management, News-Vendor Problem, Multiple discounts, Substitution, Assortment, Drop-shipping, Product returns*

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Introduction

In this chapter, we give a general introduction for the work carried out in this thesis. The objective of this chapter is: first, to provide a description of the various types of products for which the inventory modelling approach used in this thesis can be applied; second, to give details on the News-Vendor Problem (NVP) which is the inventory control problem that our work is based on; third, to describe the work done in this thesis and present our main contributions.

1.1 Background

The News-Vendor (NV) context which this thesis is based on is particularly adapted for some types of products characterized by a short life cycle, long replenishment times and/or seasonal demand patterns. Indeed, the world economy is six times larger than it was half a century ago, growing at an annual rate of 4% during the period. New technologies have paved the way for more efficient production systems in a wide range of industries and have promoted the economic growth. The rise of globalization, especially over the past two decades with the growing trade and financial integration of the world economy led to much faster diffusion of ideas and cultural products [1]. One of the most profound changes in the last decade is the dramatic shrinkage of product life cycles [2] because of the ever-increasing competition: a manufacturer faces competition from many other global companies in addition to local manufacturers and everyone offers more and more new products to the market with innovations brought by technological advances. For example, electronic products update very fast: iPhone has tens of major releases since the original one born in 2007. For fashion, apparel, luxury and other soft-good industries, the product life cycle is also very short. Zara, for instance, delivers new products twice a week to its 1,670 stores around the world. This adds up to more than 10,000 new designs each year [3].

The second important characteristic for products of interest (those for which the NVP is well suited) is the seasonal profile observed in sales. Many retail businesses see a great part of their profits generated in one or two seasons during the year, the end-of year or Christmas season being such a typical busy period. Examples of products with highly seasonal demand include: Christmas gifts, Valentine gifts, fireworks, swimwear, holidays, clothes, etc. Another obvious example of demand seasonality is the great online sale peak (in 2015 for example, 91.2 billion CNY in 24 hours) which happens on 11 November, the "singles day" in China.

These products which have short selling periods are called seasonal products compared to permanent products which are displayed in markets all the time. Seasonal products bring many challenges especially for retailers because the demand is uncertain: they need to make a purchasing order before the selling season because of the long production and/or distribution lead time compared with the short selling period; if the stock is not enough, there is a risk that there will be an underage in the selling period and a penalty cost should be paid in many situations; if the order quantity is too big, there will be depreciation at the end of the season. Managers often have to make decisions regarding the inventory level over a very limited period, this is the case, for example with seasonal products such as Christmas cards that should satisfy all demand in December, but any cards left in January have almost no value.

Retailers of seasonal products need to sell products within a short time while the needs of consumers are constantly changing. A successful retailer managing seasonal products must satisfy two requirements: to adjust for trends and to improve revenue. Three characteristics should be especially considered for such products.

- dealing with overstock (discount)
- product variety
- free product returns policy

Indeed, using discounting can permit to reduce the risk of overage for products sold in the season. Besides, product variety and assortment decision is a key factor for products offered to consumers. Furthermore, product returns is a more and more observed phenomenon in contexts such as retail e-commerce. The goal of the present thesis is to consider these three extensions in order to contribute to enhance the understanding of challenges associated with the NV inventory control problem. Our aim is to contribute to the development of models pertaining to the NVP, so as to gain useful guidelines for practitioners.

1.2 Context: the News-Vendor Problem

The NVP, also known as the single-period inventory problem or Newsboy Problem, is a classical problem in inventory management aiming at finding the optimal order quantity which maximizes the expected profit under probabilistic demand. Its name derives from the context of a NV purchasing newspapers to sell before knowing how many will be demanded that day. The optimal order quantity is deduced from the trade-off between two situations: if the order quantity is not enough, the NV loses some possible profit; on the other hand, if the order quantity is too large, overstock happens. It occurs whenever the demand is random, a decision must be made regarding the order quantity prior to finding out how much is needed, and the economic consequences of having "too much" and "too little" are known. The NVP has a long history that can be retrospected to [4] in which a variant is used to describe and solve a bank cash-flow problem. The NVP has been paid more and more attention over the past half century. The increasing attention can be explained that the NVP is applicable in many real situations: service industries [5] that have gained increased dominance, fashion and sporting industries [6], etc.

In Sect. 1.2.1, we will firstly present the basic model of NVP. In Sect. 1.2.2, we will present early achievements on NVP by dividing the extensions of the NVP into 4 categories.

1.2.1 Basic problem: classical NVP model

To solve the classical NVP, researchers have developed an approach by maximizing the expected profit. To show how this research approach works, we define the following notations. These notations will be used throughout the thesis.

x	the demand during the selling season, a random variable
f(x)	the probability density function of x
F(x)	the cumulative distribution function of x
v	unit selling price
w	unit purchasing cost
s	unit salvage value
p	unit shortage penalty
Q	order quantity, the decision variable

Since the demand is not realized before the selling season, the NV does not know the future profit. The traditional approach is based on assuming a risk neutral NV who decides the optimal order quantity before the selling season to get the maximum expected profit. The profit per period is:

$$\pi = \begin{cases} vx - wQ + s(Q-x) & \text{if } x < Q \\ vQ - wQ - p(x-Q) & \text{otherwise} \end{cases} \tag{1.1}$$

By taking the expected value of π, we get the following expected profit:

$$\begin{aligned} E(\pi) &= \int_0^Q (s-w)Qf(x)dx + \int_0^Q (v-s)xf(x)dx + \\ &\quad \int_Q^\infty (v-w+p)Qf(x)dx + \int_Q^\infty -pxf(x)dx \end{aligned} \tag{1.2}$$

By using Leibniz's rule to obtain the first and second derivatives, we show that $E(\pi)$ is strictly concave. The optimal order quantity (Q*) condition satisfies the following formula:

$$F(Q^*) = \frac{p+v-w}{p+v-s} \tag{1.3}$$

The expected profit corresponding to the optimal order quantity Q^* turns to be:

$$E(\pi(Q^*)) = (v - s)\mu - (v - s + p) \int_{Q^*}^{\inf} x f(x) dx \tag{1.4}$$

Some researchers use also a cost minimizing approach to solve the problem in terms of balancing the costs of underestimating and overestimating demand and they find same results. We use the expected profit maximizing approach in our work.

1.2.2 Early achievements

After [7] formulated the NVP, interest in the NVP remains unabated and many extensions to it have been proposed in the last decades. [8] reviewed these extensions and classified them into 11 categories: 1. Extensions to different objectives and utility functions. 2. Extensions to different supplier pricing policies. 3. Extensions to different NV pricing policies and discounting structures. 4. Extensions to random yields. 5. Extensions to different states of information about demand. 6. Extensions to constrained multi-product. 7. Extensions to multi-product with substitution. 8. Extensions to multi-echelon systems. 9. Extensions to multi-location models. 10. Extensions to models with more than one period to prepare for the selling season. 11. Other extensions. [9] extended the prior review by considering several specific extensions such as integrating marketing effort, stock dependent demand, and buyer risk profiles and how they influence the determination of the optimal NV order quantity.

These two works bring lot of convenience for future research, however, there are some extensions of NVP not included in these categories, e.g. NVP extensions considering the product assortment problem or product returns. We use a more intuitive way to classify the research works on the NVP by considering three actors (supplier, NV and consumers) and one object (product). Therefore, we can classify the different extensions developed so far into four categories as illustrated in Figure 1.1. In fact, the extensions on the NVP are based on different assumptions according to activities that can be described in these 4 categories. For example, the extension considering quantity discounts comes from the fact that suppliers often provide discounts for the NV according to the quantity he/she orders. This discount activity is operated by the supplier. The NV also uses discount to attract consumers, this activity is operated by the NV. By using this method, we provide an intuitive way to present the extensions on NVP and future extensions can find their positions in this classification.

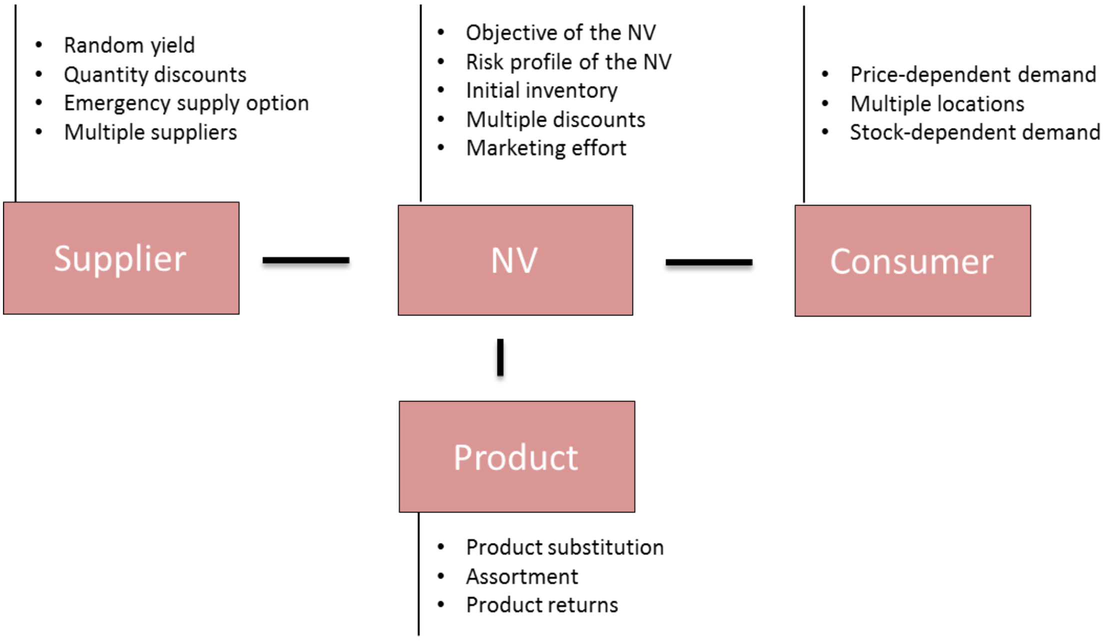


Figure 1.1: 4 categories of NVP extensions

1.2.2.1 Extensions concerning the supplier

Extensions in this category consist of random yields (the production capacity of the supplier is a random variable), quantity discounts, emergency supply option, etc, for both single- and multi-supplier cases. Some of these extensions are described below.

Random yield: [10] reviewed random yield models, and presented five basic approaches: (i) a Bernoulli process; (ii) stochastically proportional yield; (iii) stochastic yield proportional to order quantity; (iv) random capacity; and (v) general model that specifies the probability of each output for each order quantity. [11] solved the NVP under multiple suppliers with stochastic yield. [12] derived the optimal order quantity for interdependent demand and supply for a NV facing stochastic supply yield, in addition to stochastic demand. Increasing product complexity, manufacturing environment complexity and product quality all lead to uncertainties in production. [13] assumed the productive capacity is a random variable y, $f_0(y)$ is the probability density of y, and $F_0(y)$ is the cumulative distribution function of y. The planned production is Q, so the actual production is $min\{Q, y\}$. [13] proved that the expected profit is concave on order quantity and the optimal quantity is the same with the classical NVP model.

Quantity discounts: The determination of the optimal order quantity when the supplier offers quantity discounts has been treated in many NVP extensions [14, 15, 16]. There are basically three types of quantity discounts [14]: a. All-units quantity discounts (for Q such that $q_j < Q < q_{j+1}$, the cost per unit is w_j. The discount applies to all units purchased); b. Incremental quantity discounts (the discount applies only to the additional units after the break-points); c. Carload-lot discounts (any quantity in the "carload-lot" interval assesses the maximum cost). [14] showed that the behavior of a NV facing an all-units quantity discount depends on the cost of disposing of excess inventory which can be: (i) zero, (ii) negative and (iii) positive. [17] proposed algorithms for solving a NVP in which Q is made up of a number of containers with standard sizes. The NV can choose any combination of container sizes. The larger the container the smaller the unit cost. [16] considered all-units and incremental quantity discounts and dual performance measures. [15] proposed three extensions to the NVP: (1) supply of inventory is a random variable due to a supplier with variable capabilities, (2) suppliers are charged a penalty for not being able to meet contract obligations; the penalty can be fixed or proportional to the quantity of shortage and (3) a secondary supplier can supply additional units when the primary supplier can't provide Q^*. The secondary supplier charges a higher unit price.

Emergency supply option: [18] assumed that when the primary supplier can not provide Q^*, a secondary supplier can supply additional units. But only a proportion of demand can be satisfied from the emergency supply option in case of a shortage. r is the unit cost from the emergency supply option while $w < r < v + p$. [18] showed that the optimal order quantity is smaller than the optimal order quantity in the classical problem: in presence of emergency supply, some demand is not lost when there is a shortage. [19] incorporated the drop-shipping as an emergency option into the single-period model framework and showed that it can lead to a significant increase in expected profit. [20] assessed three different organizational forms that can be used when a store-based sales network coexists with a web site order network. The three organizational forms are store-picking, dedicated warehouse-picking and drop shipping. Authors used a NV type order policy model to compare the three different models and to analyze the impact of some parameters on inventory policies in the supply chain. [21] proposed a mixed mode that utilizes both traditional and drop-shipping modes for seasonal fashion and textiles chains, in order to take full advantage of demand fluctuation and improve the profit-making ability.

Multiple suppliers: [22] studied a supplier selection problem, where a buyer, while facing random demand, is to decide ordering quantities from a set of suppliers with different yields and prices. [23] considered the problem of a NV that is served by multiple suppliers, where any given supplier is defined to be either perfectly reliable or unreliable. [24] addressed the supplier selection and purchase problem with fixed selection cost and limitation on minimum and maximum order sizes under stochastic demand.

1.2.2.2 Extensions concerning the NV

Extensions in this category consist of different objectives and profiles of the NV, initial inventory, multiples discounts and marketing effort. Some of these extensions are described below.

The NV with other objectives: Besides the objective to maximize the expected profit or minimize the expected cost, some researchers consider the maximization of the probability of achieving a target profit [25, 26, 27]. They suggested that maximizing the probability of achieving a target profit level is a realistic managerial objective in the NVP.

Risk profile: The NV can have various risk preferences including, risk-neutral, risk-averse and risk-seeking preferences. Alternative risk preferences such as loss-aversion, have also been analyzed in the context of the NVP. [28] provided a detailed investigation of the effects of risk, risk aversion and changes in various price and cost parameters for a risk-averse retailer. [29] investigated the pricing, ordering and promotion policies of a risk-sensitive (risk-averse or risk-seeking) NV under price-dependent and stochastic demand. [30] examined the ordering policy of a loss-averse NV.

Initial inventory: This situation occurs in practice when there is an initial stock I or a stock of convertible units that can be transformed into end items [31, 32, 33]. [32] showed that expected profit is concave in I and Q and that there is a critical level of I above which no order will be placed under certain yield, and this level is the same under random yield.

Multiple discounts: It happens frequently in practice that multiple discounts are progressively used to sell excess inventory. Multiple discounts are especially common in the apparel industry where discounts get steeper as the season draws to an end. [27] solved a NVP with multiple discounts with these assumptions: every discount results an additional demand, which is proportional to the original demand; the remaining inventory can be sold at the final discount. [27] proved that for the NVP under progressive multiple discounts, the expected profit is concave and developed the optimality condition.

Marketing effort: The assumption is that the demand is influenced by marketing effort (e.g. advertising). An increase in mean demand due to marketing effort leads to an increase in the optimal stocking quantity Q^*, but it is not so clear for the impact of an increase in demand variability. [34] proved that the optimal marketing effort can be determined by the following formula, where C is the unit cost of effort: $(v-w)\frac{du}{de} - \frac{dc}{de} = 0$. The analysis presented is extended to a situation where marketing effort affects demand in a way that demand variance decreases as more effort is made in the selling season. [35] examined the effects of demand randomness on optimal order quantities and the associated expected costs by applying mean-preserving transformations to the demand variable.

1.2.2.3 Extensions concerning consumers

Price-dependent demand: The demand can be influenced by the selling price. Extensions on this subject give some basic price-demand relationship assumptions. The linear and multiplicative relationships are the basic ones.

In the classic NVP, the selling price is considered as exogenous, over which the retailer has no control. This is true in a perfectly competitive market where buyers are mere pricetakers. However, retailers may adjust the current selling price in order to increase or decrease demand. Therefore, several researchers have suggested extensions of NVP in which demand is assumed to be price dependent. [36] assumed that price-dependent demand is affected additively by a random variable, which is independent of the selling price. [37] introduce the case of a multiplicative model in which the stochastic demand is affected multiplicatively by a random variable. Price-dependent demand NVP has then been largely studied [26, 31, 38, 39, 40, 41].

Location: Multi-location NVP extensions can be divided into two types: (1) all locations have the same selling season and (2) the selling seasons of the different locations lag each other. [42] analyzed the effects of centralization on the multi-location NVP. In this model, there are n retail centers which raises the opportunity for centralization. [42] compared the expected cost of two configurations: (a) a decentralized system in which a separate inventory is kept at each center and (b) a centralized system in which inventory is kept at central warehouse. [42] assumed normal demand distribution and linear holding and penalty costs and showed that the expected cost of the decentralized facilities exceeds that of the centralized facility with the difference depending on the correlation of demands. For uncorrelated and identically distributed demands, the expected cost of the centralized facility increases as the square root of the number of consolidated centers. [43] considered the situation where a NV exploits the difference in timing of selling seasons of geographically dispersed markets. For example, a US garment maker can sell his/her remaining summer fashion in Australia where summer is about to begin. [43] treated both centralized and decentralized case.

Stock-dependent demand: [44] was the first to consider stochastic demand when inventories stimulate demand within a single-product, single-period setting. [45] developed a stochastic model that jointly optimized inventory and price and captured the effects of a store's fill-rate on consumer utility. [46] proposed a more general, stochastic demand modeling framework that encapsulates the influence of inventory on the demand distribution. They provided insights on the optimal inventory policy of a single product when price is also a decision variable. [47] employed the same modeling framework to capture the dependence of demand on inventory in a stochastic setting and extended it to the case of two products under product substitution.

1.2.2.4 Extensions concerning products

In the real situation, it is not usual for a retailer to sell only one product. Two products or even multiple products could be involved in the business. With multiple products, the NV needs to consider the substitution effect (some consumers preferring one product which is out of stock could buy other products for substitution) and to decide which products to sell in the selling season. In addition, product return is also an important issue for retailers to considering when they are making decisions. Here are some related extensions on NVP.

Substitution: The topic of product substitution in inventory management first appears in [48]. Papers on this topic can be divided into 3 categories according to the substitution type: papers of the first category deal with one-direction substitution or firm-driven substitution, where only a higher grade (e.f. quality, size, etc.) product can substitute a lower grade product, when the supplier makes decisions for consumers on choosing substitutes (see, e.g., [49, 50, 51, 52, 53]). The second category consists of papers where arriving consumers' number follows a stochastic function and consumers make purchasing decisions under probabilistic substitution when their preferred product is out of stock (see, e.g., [54] and [55]). The third category consists of papers considering that each product can substitute for other products and the fraction that one out-of-stock product is substituted by another product is deterministic (see[48, 56, 57, 58, 59, 60, 61, 62, 63, 64]). [61] obtained optimality conditions for both competitive and centralized versions of the single period multi-product inventory problem with substitution.

Assortment and substitution: Assortment planning in the area of NVP has been extensively studied too. [65] made a comprehensive review of the recent literature. In some papers, the substitution effect and the assortment planning are simultaneously considered. Two major types of demand modelling were used in earlier achievements: utility maximization (see [55, 66, 67]) and exogenous demand models (see [54, 68]). [66] considered a static substitution model with multinomial logit (MNL) demand distributions assuming that consumers are rational utility maximizers. They show that in this model the optimal solution consists of the most popular product. [55] studied a joint assortment and inventory planning problem with stochastic demands under dynamic substitution (assuming that a consumer's choice is made from stock on hand) and general preferences where each product type has per-unit revenue and cost, and the goal is to maximize the expected profit. Assuming that consumer sequences can be sampled, they propose a sample path gradient-based algorithm, and show that under fairly general conditions it converges to a local maximum. [67] consider a single-period joint assortment and inventory planning problem under dynamic substitution with stochastic demands, and provide complexity and algorithmic results as well as insightful structural characterizations of near-optimal solutions for important variants of the problem.

Product returns: In the literature, consumer returns are typically assumed to be a proportion of products sold (e.g.[69, 70, 71, 72, 73]), which obviously implied that if more items are sold, more products will be returned from consumers. [74] empirically showed that the amount of returned products has a strong linear relationship with the amount of products sold. Based on the assumption that a fixed percentage of sold products will be returned and that products can be resold at most once in a single period, [70] investigated optimization of order quantities for a NV style problem in which the retail price is exogenous. [75] considered a manufacturer and a retailer supply chain in which the retailer faces consumer returns. [76] also assumed that a portion of sold products would be returned and discussed the coordination issue of a one manufacturer and one retailer's supply chain. [73] examined the pricing strategy in a competitive environment with product returns. [77] considered consumer return for retailer who is confronted with two kinds of demand: one needs immediate delivery after placing an order and the other accept delayed shipment, and a NV model with resalable returns and an additional order is developed. However, the model was under assumption that total demand distribution is given and each kind of demand presents a proportion of the total demand and concavity is not proved.

1.2.3 Motivations

Although lots of work have been done in the NVP area, interest in the NVP is still important. The literature of NVP has seen a big rise in the last decade. As economic activities are showing new tendencies, e.g. international cooperation and e-commerce, retailers are facing new situations. As a result, the literature of the NVP needs to be enriched. In the following, we highlight some motivations with regard to models we develop in this thesis.

Our models aim at solving problems encountered in practice within a NV framework. Multiple discounts, product variety and e-commerce (i.e. drop-shipping and product returns) are three important issues that we consider.

First, we are inspired by the fact that most retailers use several discounts to sell excess inventory. In this situation demand depends on product selling price and discounts are a certain percentage of the initial selling price. Indeed, in many situations, demand depends on product's selling price since demand would increase when selling price decreases. This relationship enables retailers to adjust the selling price to influence demand. In chapter 2, we consider this problem and assume that demand is price dependent. Two special demand-price relations are considered: additive and multiplicative cases. The motivation for the assumption of multiple discounts is reported in [27]. In realistic situations, multiple discounts are progressively used to sell excess inventory that, in turn, impact demand. This is for instance encountered in the apparel industry where the initial selling price has an important influence on demand realized during the regular selling period and discounts get deeper as the season draws to the end.

Second, product variety is another key element that is interesting to analyze in a NV context. Demand for variety comes both from the taste of diversity for an individual consumer and diversity in tastes for different consumers. However, despite the advantages of product variety, the full range of variety cannot be supplied generally, owning to the increase in inventory, shipping, and merchandise presentation (i.e. product display cost), etc. Within this context, the optimization of product assortment (i.e. products that will be offered for purchase within the store), and the optimal order quantity for each product, is a relevant decision that retailers face. By considering multiple products, two important factors should be considered to optimize the assortment and the optimal order quantities. First, product variety brings possible substitution when underage happens: the different variants of the same product (variants are products of different colors for instance), may act as substitutes when the consumer finds that a product is out of stock. Second, besides the purchasing cost which increases with the order quantity, there is a fixed cost associated with each variant of product included in the assortment, e.g. the material handling and merchandise presentation cost stemming mainly from the space and labor cost required to display products in the store, etc. Joint assortment planning and inventory management problems with substitution have been extensively studied in the literature [65]. However, some limits exist in earlier works, e.g. [54] assumed that the order quantities are set to achieve a fixed service level and give two bounds of the product demand. The final results are based on the approximation of the demand and the numerical examples are mainly in the case of items with uniform market share. We consider two effects in a multi-product NVP: the transfer of demand owning to the unlistment of some products, then the substitution between products included in the assortment and give the formulation of the expected total profit.

Third, e-commerce activity along with the drop-shipping option is another variant that we analyze. In recent years, retailers have used the drop-shipping mode as an order fulfilment strategy. Drop shipping is especially interesting for seasonal products. Seasonal products have short selling season and long lead replenishment time thus the order is placed to a faraway supplier before the selling season and it is not possible to place another one during the season when the retailer finds that the product is out of stock. Then drop shipping can be used to fulfill this part of demand. We assume a mixed strategy to satisfy demand: use both store inventory and drop shipping option. The motivation of this assumption is reported in [19]. A disadvantage of e-commerce is that product returns are especially problematic: products sold through e-commerce tend to have higher return rate than traditional process [70]. As the classical NVP, store demand (the demand of consumers shopping physically in the store) is satisfied by store inventory. The NV can use the store inventory to satisfy internet demand and has in addition a drop shipping option for internet demand. When products are delivered, some consumers are unsatisfied and then a portion of products is returned to the store.

1.3 Description of the manuscript and main contributions

This thesis consists of 3 main parts. Those three parts deal with the inventory management for a NV by focusing on specific points. Chapter 2 addresses particularly the pricing and overstock issues by introducing multiple discounts. Chapter 3 rather focuses on the assortment planning problem by considering the substitution effect for a NV who provides multiple products. Chapter 4 focuses on the mixed supplying strategy for a NV who uses drop-shipping to satisfy Internet demand and has a free return policy. Those chapters are all organized in the same way: introduction, related literature revue, modeling, formulation of the model, numerical results and conclusion.

In more details, Chapter 2 considers a NVP with multiple discounts that are used progressively after the regular selling season. The demand is price dependent and the NV decides the initial selling price to make a maximal profit. As we know, the optimal initial price is affected by the discount scheme (which consists of discount frequency and discount percentages), and the initial price itself affects the optimal inventory decision. Therefore, we analyze the joint determination of optimal order quantity, optimal initial selling price and optimal discount scheme. Firstly, we prove the concavity of the expected profit in function of the order quantity. We develop a general expression of the optimal order quantity for both the additive and multiplicative price-dependent demand cases with general demand distributions and provide a simple expression of the expected profit corresponding to the optimal order quantity. In addition, these expected profit equations show a much clearer insight into the impact of initial price and discount number on the expected profit. Approximate functions for the expected profit are derived. Numerical examples show that at a given initial price, the expected profit increases with the discount number, but it has an upper bound. It is not reasonable to use too many discounts, because the increasing speed of the expected profit decreases and tends to be zero. For additive demand, the expected profit is approaching the maximum value with the linear discount scheme and with the exponentially declining scheme for multiplicative demand. Numerical examples show also that the approximate functions provide accurate results.

In Chapter 3, we extend the classical NVP to consider the assortment and substitution effects. We develop a model considering demand transfer and demand substitution. The transfer and substitution fractions are formulated. Then, a random-walk Monte Carlo method provides an efficient computational approach to get the value of the expected optimal profit and optimal order quantities for a product assortment. Numerical examples show insights regarding the performances of the NVP. Our examples indicate that demand transfer and substitution have important impacts on the assortment, expected profit, and optimal order quantities. With a global optimization policy, several results can be derived from numerical results: the expected profit decreases with the fixed cost value, the fraction of lost sale and demand uncertainty. Assortment size increases with the fraction of lost sale but decreases with the fixed cost value. The model can easily be adapted to problems with other kinds of substitution such as one-item substitution, which can be treated in the same way by our model only changing the demand transfer and substitution equations.

In Chapter 4, we consider a NVP with drop-shipping option to satisfy demand. Many retailers use a mixed drop-shipping and store inventory strategy to satisfy demand. In this chapter we formulate a NV model for inventory management of a mixed supplying strategy considering different return rates for different kinds of delivery: store inventory to store demand, drop-shipping for internet demand and store inventory to internet demand. We provide the optimal solution for store order quantity under general demand distributions and the expression of the corresponding expected profit. In a situation where the return rate of drop-shipping is higher than the one of store inventory to internet demand delivery, the expected profit is proved to be a concave function of the store order quantity under a reasonable condition. Our examples indicated a high reliance on store inventory for the NV and thus it is not reasonable for the e-retailer to use only drop-shipping option and the higher is the return rate related to drop-shipping option, the higher is the reliance on store inventory.

At the end of the manuscript, we close the thesis by giving general concluding remarks and highlighting directions for future research.

The work associated with Chapter 2 was presented on the 5th International Conference on Information Systems Logistics and Supply Chain held at the Castle of Breda, Netherlands. We have submitted it to "Journal of Industrial and Management Optimization". The work of Chapter 3 was presented on the International Conference on Industrial Engineering and Systems Management held at Seville, Spain and has been submitted to "OR Spectrum". The work of Chapter 4 has been submitted to "International Journal of Production Research".

2

NVP with price-dependent demand and multiple discounts

Existing papers on the NVP that deal with price dependent demand and multiple discounts often analyze those two problems separately. This chapter considers a setting where price dependence and multiple discounts are observed simultaneously, as is the case of the apparel industry. Henceforth, we analyze the optimal order quantity, initial selling price and discount scheme in the NVP context. The term "discount scheme" is often used to specify the number of discounts as well as the discount percentages. We present a solution procedure of the problem with general demand distributions and two types of price-dependent demand: additive and multiplicative case. We provide interesting insights based on a numerical study. An approximation method is proposed which confirms our numerical results.

2.1 Introduction

Pricing and multiple discounts are common features observed in real life NVP. In many situations, demand depends on product's selling price since demand would increase when selling price decreases. This relationship enables retailers to adjust the selling price to influence demand. Furthermore, multiple discounts mean that the retailer uses a certain number of discounts to sell excess inventory, rather than performing only one discount. In realistic situations, multiple discounts are progressively used to sell excess inventory that, in turn, impact demand. This is often encountered in the apparel industry where the initial selling price has an important influence on demand realized during the regular selling period and discounts get deeper as the season draws to the end. This end of season, for example, is called the discount period in France, which happens twice every year.

The work we carry in this chapter is motivated by the fact that most retailers use several discounts to sell excess inventory. In this situation demand depends on the selling price and discounts are a certain percentage of the initial selling price. The term "discount scheme" is often used to specify the number of discounts as well as the discount percentages. A special discount scheme where the discount prices are equally spaced, is called a linear discount scheme. In this work, given the unit purchasing cost, salvage value and the price-demand relationship, we are concentrating on the determination of the order quantity, the initial selling price and the discount scheme that would maximize the expected profit. Two special demand-price relations are considered: additive and multiplicative cases. In the additive case, the mean demand decreases linearly with the selling price, while in multiplicative case, the mean demand decreases exponentially. These two relations are common expressions used to represent the price-dependent demand in practice [9]. [27] obtains the optimality condition of the order quantity for a NV considering multiple discounts. [78] extends to the case where multiple discounts are used and the demand is price-dependent. The concavity is proved for the NVP with uniformly distributed demand, the condition of optimal order quantity is obtained while the discount prices are linear and the demand-price relationship is considered to be additive.

This chapter extends the work of [78] since: (1) we demonstrate the concavity for the NVP with multiple discounts and price-dependent demand under general demand distributions and obtain the optimality condition of the order quantity, i.e. the concavity is not limited to uniform distribution; (2) we provide a simple expression of the optimal expected profit; (3) we obtain optimality conditions of the order quantity for both additive and multiplicative demand case; (4) based on a numerical study, we show some new insights, e.g. on the optimal discount scheme; (5) under some conditions we write the expected profit function in a manner that enables to search the numerical optimal initial selling price. This approximation method confirms the insights observed in numerical studies.

The rest of this chapter is organized as follows. Section 2.2 presents the literature review related to the work we carry in this chapter. In section 2.3, we formulate the multiple discounts and price-dependent NVP. In Section 2.4, we solve the order quantity and initial pricing decisions with the objective of maximizing the expected profit, for additive price-dependent demand. Numerical examples are then provided. In Section 2.5, we treat the case of multiplicative demand in the same way. Section 2.6 contains further discussions and some suggestions for future research.

2.2 Literature review

Interest in price-dependent and multiple discounts problem goes on in the last decades. One of the latest work is [79] who consider the price-dependent and multiple discounts problem with multiple periods over a product's life. [79] review works on price-dependent and multiple discounts problem, but they are not focused on the NVP. Therefore, we review the earlier achievements in the area of NVP, which consists of two streams, i.e.: (1) the NVP with price-dependent demand and (2) the NVP with multiple discounts.

In the classic NVP, the selling price is considered as exogenous, over which the retailer has no control. This is true in a perfectly competitive market where buyers are merely pricetakers. However, retailers may adjust the current selling price in order to increase or decrease demand. Therefore, several researchers have suggested extensions of NVP in which demand is assumed to be price dependent. [36] assumes that price-dependent demand is affected additively by a random variable, which is independent of the selling price. [37] introduce the case of a multiplicative model in which the stochastic demand is affected multiplicatively by a random variable. [26] examine the pricing and ordering policies of a NV facing a random price-dependent demand under two different objectives, (1) the objective of maximizing the expected profit and (2) the objective of maximizing the probability of achieving a certain profit level. Analytical solutions are obtained for the additive price-demand relationship with normal distribution. They develop numerical procedures for another case of demand: the demand distribution is constructed using a combination of statistical data analysis and experts' subjective estimates. [31] investigates the joint pricing and ordering decisions under general demand uncertainty, aiming to reveal the fundamental properties independent of demand pattern. Unimodality of the expected profit function that traces the best price trajectory over the order-up-to decision was proved under the assumptions that the mean demand is a monotone decreasing function of price. [38] investigate the price-dependent NV model in a competitive environment. They show the conditions for the existence of the pure-strategy Nash equilibrium and its uniqueness. [39] introduces a price-dependent demand with stochastic selling price into the classical NV, analyses the expected average profit for a general distribution function of price and obtains an optimal order quantity. [40] studies the channel coordination with a return policy that lets the manufacturer share the risk of demand uncertainty. The manufacturer's decision is to identify both the optimal wholesale price and the return policy, based on the retailer's reaction on that offer. The retailer in turn optimizes the retail price and the order quantity to meet a price-dependent uncertain demand. [41] develops a distribution free approach to NVP with price-dependent demand for the situations in which the NV may be missing demand distribution information or historical demand data may not fit any standard probability distributions. Lower bounds on the expected profit are shown to be jointly concave in price and order quantity.

[27] solves a NVP in which multiple discounts are used to sell excess inventory. In this model, retailers progressively increase the number of discounts until all excess inventories are sold out. The product is initially sold at a regular price v_0. After some time, if any inventories remain, the unit price is reduced to v_1, $v_0 > v_1$. Then, a second discount with a selling price $v_2 (v_1 > v_2)$ is made, etc. The amount demanded for each value of v_i is assumed to be a multiple of the demand realized at the regular selling price and moreover, the coefficients are assumed to be given. [27] solves the problem under two objectives: (a) maximizing the expected profit and (b) maximizing the probability of achieving a target profit. [27] shows that the expected profit is concave and derived the sufficient optimality condition for the order quantity. A closed-form expression for the optimal order quantity is obtained for the objective of maximizing the probability of achieving a target profit. [80] develops an algorithm for identifying the optimal order quantity for the multi-discount NVP when the supplier offers the NV an all-units quantity discount. [81] provide a solution algorithm to the multi-product multi-discount constrained NVP. [78] extends the NVP to the case where demand is additively price dependent and multiple discount prices are used to sell excess inventory. Given the initial price and linear discount scheme, he solved the condition of the order quantity

which maximizes the expected profit prior to any demand being realized. [82] consider an inventory problem for gradually obsolescent products with price-dependent demand and multiple discounts. They assume that the increase of demand due to price change is linearly correlated with the difference between prior and present prices. However, the demand is assumed to be deterministic as a function of time, which is a limited assumption for the NVP context.

Our work focuses on the NVP and differs from previous works according to the different points summarized in Table 2.1. We generalize the NVP with multiple discounts in three aspects: price-demand relation, demand distribution and discount scheme.

parameter	[27]	[78]	our work
price-demand relation	fixed	additive	additive and multiplicative
demand distribution	general	uniform and normal	general
discount prices	known	linear	all types (linear and non-linear)

Table 2.1: Comparison with the work of Khouja (1995,2000)

2.3 The problem under study

Figure 2.1 represents the sequence of events in a selling season. A season consists of n+1 sub-periods where each sub-period i (i=0,...,n) is characterized by a unit selling price and a stochastic demand which depends on the selling price offered to customers during the sub-period. At the beginning of the season, the NV buys from the supplier a quantity Q of products at unit price w. This quantity has to cover all demand during the selling season since we assume in this model that the NV can not buy products during the season. In sub-period i=0, i.e. the regular selling period, the product unit selling price is v_0, the random demand is X_0 and the realization of X_0 is x_0. In sub-period i=1, i.e. the first discount period, the product unit selling price is v_1, the total demand until the end of this period (including X_0) is X_1, and x_1 is the realization of X_1. The rest of periods can be deduced in the same way. As selling season goes on, the discounts get deeper and the NV captures some additional demand in each discount period, until the final discount period, i.e. sub-period i=n, where all remaining products are disposed of at a unit price s where $s = v_n$. These discount prices are not given, but for a linear scheme, the discount prices are equally spaced between v_0 and s. Otherwise, we call it a non-linear scheme.

The objective of our problem is to find the order quantity Q that maximizes the expected profit.

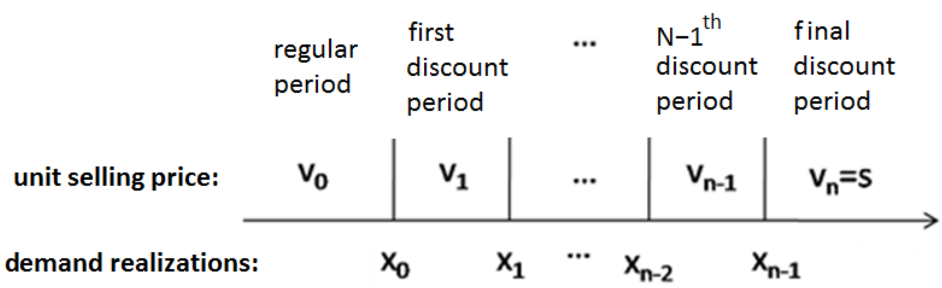


Figure 2.1: Sequence of events for a selling season

Define the following notations used in Chapter 2:

X_0	Demand during the regular period with mean μ_0 and standard deviation σ_0
x_0	Realization of X_0
f	Density function of X_0
F	Cumulative distribution of X_0
$X_i(i > 0)$	Demand accumulated till the sub-period i, with mean μ_i and standard deviation σ_i, $\mu_n = \infty$ (all products are disposed of with s)
$x_i(i > 0)$	Realization of X_i

Given variables:

s	Salvage price per unit, $s = v_n$
w	Purchase price per unit

Decision variables:

v_0	Regular selling price (initial price) per unit,
n	The number of discounts during the season
v_i	Unit selling price at the i-th discount period, $v_0 > v_1 > \cdots > v_i > \cdots > v_n$
Q	Order quantity

The random profit function is a multivariate function of selling prices and the order quantity:

$$\pi(Q)=\begin{cases} v_0Q-wQ & x_0>Q \\ v_0x_0+(Q-x_0)v_1-wQ & x_0\leq Q<x_1 \\ v_0x_0+(x_1-x_0)v_1+(Q-x_1)v_2-wQ & x_1\leq Q<x_2 \\ \vdots \\ v_0x_0+(x_1-x_0)v_1+\cdots+(x_{i-1}-x_{i-2})v_{i-1}+(Q-x_{i-1})v_i-wQ & x_{i-1}\leq Q<x_i \\ \vdots \\ v_0x_0+(x_1-x_0)v_1+\cdots+(x_{n-1}-x_{n-2})v_{n-1}+(Q-x_{n-1})v_n-wQ & x_{n-1}\leq Q \end{cases} \tag{2.0}$$

The profit related to the interval: $x_{i-1}<Q<x_i$, is the sum of the revenue of the regular period v_0x_0, the first i-1 periods $(x_1-x_0)v_1+\cdots+(x_{i-1}-x_{i-2})v_{i-1}$, and the i-th period $(Q-x_{i-1})v_i$, subtracted by the total purchase cost wQ.

Let us remark that the quantity demanded at the i-th discount period is a function of the quantity demanded in the first (regular) period and the selling price associated with the i-th discount. This function depends on how the price-demand relation is modeled. Hence two cases are considered, the additive price-dependent demand (cf. Section 2.4) and the multiplicative price-dependent demand (cf. Section 2.5).

2.4 Optimal pricing and ordering decisions for additive price-dependent demand

In the case of additive price-dependent demand, the mean demand μ decreases linearly with the price v, i.e., $\mu=a-bv$, a and b are both positive constants obtained from historical data. [78] assumes that $v=W-Bx$, where B is a positive constant known to the NV (it equals to $1/b$ in our model), and W is a random variable with a known probability distribution whose realization becomes known only after ordering. At the end of the regular period, x_0 becomes known, thus W can be calculated by: $W=v_0+\frac{x_0}{b}$. We refer readers for more details to [78], which has considered the additive price-dependent demand case, as explained in the literature review section. Then $x_i=(W-v_i)b$ can be written as:

$$x_i=x_0+\mu_i-\mu_0 \tag{2.1}$$

2.4.1 Optimal expected profit and optimal order quantity

If we replace x_i in the profit function $\pi(Q)$ by x_0 (equation 2.0), we can derive the expected profit function $E(\pi(Q))$ (c.f. Appendix 1).

The expected profit can be developed to (c.f. Appendix 1):

$$\begin{aligned} E(\pi(Q)) \ = & \\ & Q[-w + v_0 + \sum_{i=0}^{n-1}(v_{i+1} - v_i)F(Q + \mu_0 - \mu_i)] + \\ & \sum_{i=0}^{n-1}\int_0^{Q+u_0-u_i}(v_i - v_{i+1})(x + u_i - u_0)f(x)dx \end{aligned} \tag{2.2}$$

Lemma 1. *The expected profit function $E(\pi(Q))$ is concave.*

Proof. The proof is provided in Appendix 2. □

The condition of the optimal order quantity is given by:

$$\sum_{i=0}^{n-1}(v_i - v_{i+1})F[Q^* + \mu_0 - \mu_i] - v_0 + w = 0 \tag{2.3}$$

When n=1, we get the optimality condition for the classical NVP:

$$F(Q*) = \frac{v_0 - w}{v_0 - s} \tag{2.4}$$

According to equation 2.3, the first term of equation 2.2 is zero for Q^*. So, we have:

$$E(\pi(Q^*)) = \sum_{i=0}^{n-1}\int_0^{Q^*+\mu_0-\mu_i}(v_i - v_{i+1})(x + \mu_i - \mu_0)f(x)dx \tag{2.5}$$

Equation 2.5 gives the optimal expected profit. We note that when n=1, we get the profit associated with the classical NVP:

$$E(\pi(Q^*)) = \int_0^{Q^*}(v_0 - s)xf(x)dx \tag{2.6}$$

2.4.2 Numerical Analysis

We use normally distributed demand (e.g. [78]) in our examples. Other demand distributions will also work. The concavity enables us to search the optimal order quantity by using a Golden Section method (The golden section search is a technique for finding the extremum of a strictly unimodal function by successively narrowing the range of values inside which the extremum is known to exist). Given a discount scheme, we obtain the expected profit for an arbitrary value of the initial selling price by equation 2.5. Thus we can search the optimal initial price by numerical global optimization methods.

Consider a practical example: A supplier provides a new type of T-shirt at a price $w = 3$ Euros per piece. The amount of demand(X_0) has a normal distribution N(μ_0, σ_0), the mean μ_0 will decrease linearly with the price(v_0): $\mu_0 = a - bv_0$. T-shirts can be disposed of at the end of the selling season with a price s. A manager finds that multiple discounts can improve the profit. The problem is: before the selling season, he needs to determine the order quantity, the initial selling price and discount scheme in order to maximize the profit.

2.4.2.1 First case: Linear discount scheme

$v_i = \alpha_i * v_0$, and in the linear discount case, $\alpha_i = 1 - \frac{1-\frac{s}{v_0}}{n} i (i = 1, ..., 5)$. Consider $\sigma_0 = 0$ (deterministic distribution), 2, 4, 6, 8; n increases from 2 to 21, $a = 80$, $b = 8$, and $s = 2$. By setting $v_0 = 8$, we have $\mu_0 = 16$. Figure 2.2 shows $E(\pi(Q^*))$ as a function of n.

The graph shows that the expected profit $E(\pi(Q^*))$ increases with the discount number n (with $\sigma = 2$, the expected profit is improved by about 100% with $n = 5$ compared with the classical case $n = 1$), but the increase speed is decreasing and tends to be 0 when $n \to \infty$. The reason for this result is that when $n > 1$, the NV has more opportunity to sell more products at unit price bigger than s and the opportunity tends to a limit when $n \to \infty$. We find that the expected profit decreases with σ_0. This is reasonable because for the classical NVP with normally distributed demand, the expected profit decreases with the uncertainty too. We repeated computations similar to figure 2.2 for many different combinations of $w(w \in [2, 4])$, $s(s \in [1, 3])$, $a(a \in [60, 100])$, $b(b \in [6, 12])$, $v_0(v_0 \in [6, 12])$, and similar results are obtained.

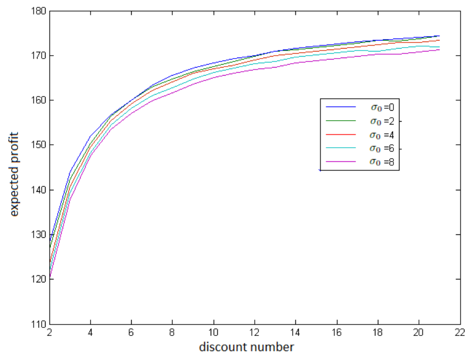


Figure 2.2: Expected profit $E(\pi(Q^*))$, as a function of the discount number, for normally distributed demand

In real life, the value of n would be limited. So we consider $n = 5$ in our analysis. Then v_0 changes from 7 to 11. For each value of v_0, equation 2.5 gives the related expected profit value. Figure 2.3 is the computing graph of the expected profit.

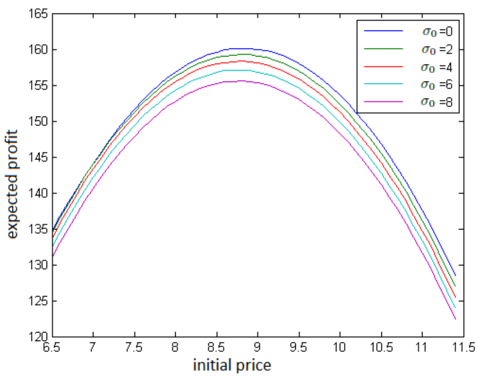


Figure 2.3: Expected profit $E(\pi(Q^*))$, as a function of the intial price

The graph shows that for the different $\sigma_0 = 0, 2, 4, 6, 8$, the expected profit is concave, thus we can derive the optimal value of the initial price from the graph. Similar results are got repeating the computations with different combinations of w, s, a, b, n.

σ_0 reflects the degree of uncertainty in demand forecast and according to $\mu_0 = a - bv_0$, b's magnitude reflects demand's sensitivity to price. The value of σ_0 and μ_0 determine the probability function $f(x)$. Using equation 2.5, table 2.2 gives the values of v_0^*, Q^* and $E(\pi(Q^*, v_0^*))$ for various combinations of b, σ_0 and n. v_0^*, Q^* and $E(\pi(Q^*, v_0^*))$ all increase with n; the effects of b follow intuitive expectation too: a lower value of b enables the firm to set a higher price, have a larger quantity of products, and realize a higher expected profit. When the uncertainty increases, $E(\pi(Q^*, v_0^*))$ decreases. This reflects the potential value of reducing demand uncertainty.

test	n	b	σ_0	v_0^*	Q^*	$E(\pi(Q^*, v_0^*))$
1	4	6	2	10.20	55.8	249.0
2	4	6	4	10.18	55.9	246.9
3	4	6	6	10.24	56.1	245.0
4	4	6	8	10.23	56.9	243.4
5	4	8	2	8.54	50.4	153.3
6	4	8	4	8.58	49.8	151.6
7	4	8	6	8.59	49.6	150.2
8	4	8	8	8.57	50.0	148.6
9	4	10	2	6.60	46.3	95.0
10	4	10	4	6.64	44.5	94.3
11	4	10	6	6.64	44.3	93.6
12	4	10	8	6.61	44.6	92.2
13	5	6	2	11.41	56.6	263.9
14	5	6	4	11.51	56.4	262.0
15	5	6	6	11.47	56.7	260.2
16	5	6	8	11.54	57.4	258.2
17	5	8	2	8.81	51.9	159.8
18	5	8	4	8.71	50.9	158.6
19	5	8	6	8.75	50.8	157.4
20	5	8	8	8.81	51.2	155.8
21	5	10	2	7.09	45.7	100.1
22	5	10	4	7.06	45.0	99.8
23	5	10	6	7.01	45.1	98.8
24	5	10	8	7.09	45.3	97.6
25	6	6	2	11.90	57.6	271.5
26	6	6	4	11.90	57.2	270.0
27	6	6	6	11.88	57.5	268.3
28	6	6	8	12.0	58.2	266.3
29	6	8	2	8.91	52.6	164.5
30	6	8	4	8.91	51.5	163.7
31	6	8	6	8.94	51.6	162.6
32	6	8	8	8.91	52.1	161.0
33	6	10	2	7.16	44.8	103.8
34	6	10	4	7.18	45.7	103.3
35	6	10	6	7.19	45.8	102.3
36	6	10	8	7.18	46.1	100.0

Table 2.2: The optimal order initial price, order quantity and expected profit for different combinations of n,b,σ_0 for normally distributed demand

2.4.2.2 Second case: Non-linear discount scheme

Consider the numerical example $n = 5, \sigma = 4, w = 3, s = 2, a = 80, b = 8$. The discount prices were produced as: $\alpha_1 v_0$, $\alpha_2 v_0$, $\alpha_3 v_0$, $\alpha_4 v_0$, s. In the linear case, $\alpha_i = 1 - \frac{1-\frac{s}{v_0}}{n} i (i = 1, ..., 5)$. Then α_i is generated by adding a term to these proportions for non-linear cases. We produce a series of discount scheme produced with a certain logic in order to asses the sensitivity of the linear discount scheme.

$\alpha_i = 1 - \frac{1-\frac{s}{v_0}}{n} i + coe(5-i)i (i = 1, ..., 5)$. We change the coefficient coe to control the perturbation of the linear discount scheme. We show here 7 series of discounts (coe=-0.03, -0.02, -0.01, 0, 0.01, 0.02, 0.03) (Figure2.4), and compute the optimal expected profit (Table 2.3).

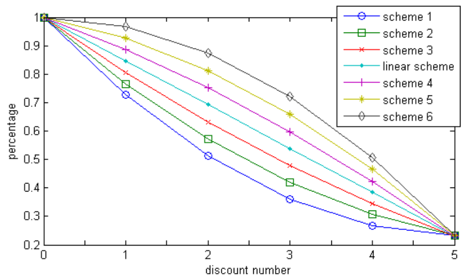


Figure 2.4: Discount schemes

scheme	coe	optimal expected profit
linear	0	158.5
1	-0.03	144.9
2	-0.02	151.1
3	-0.01	155.8
4	0.01	159.1
5	0.02	157.8
6	0.03	153.4

Table 2.3: Optimal epected profit for different discount schemes

The first line in table 2.3 is the linear case. Others are non-linear. For $coe > 0$, when coe is larger, the discount scheme curve is farther from the linear discount line and we found that the maximum expected profit decreases. The optimal initial price tends to decrease two. For $coe < 0$, when coe decreases, we find the same properties.

And when the *coe* has the same absolute value, the positive one lead to bigger expected profit,e.g. the expected profit of scheme 5 is bigger than that of scheme 2. In our cases, the linear scheme gives almost the largest expected profit, but the expected profit of scheme 4 is a little bigger. The extreme case of non-linearity is the case where all the first four discounts are 100% or the same to s. This is the same to the case that only one discount s happens: the classical NVP.

To summarize, after the discount number is fixed, it is more profitable to cut down the price slowly at the beginning of the season and then at a faster magnitude at the end of the selling season. The linear discount scheme brings an expected profit which is very close to the best one.

2.4.3 Approximation of the optimal expected profit and condition for the optimal initial price

The above numerical examples show some interesting properties, e.g. the expected profit seems to be a parabola function of the initial price and the optimal initial prices for different demand uncertainties are close to each other, see figure 2.3. However, they are not obvious to be explained from equation 2.5. An approximation method is proposed in order to explain them and it provides a faster way for the NV to make decisions. We write equation 2.5 in another way, by two steps. In the first step we consider the deterministic demand case. In the second step, we introduce the impact of the uncertainty of demand. The equation of the expected profit can therefore be decomposed in 2 components:

$$E(\pi(Q^*)) = E_\sigma + E_v + \epsilon \tag{2.7}$$

E_σ is a part of expected profit depending on σ only, E_v is a part of expected profit depending on the prices the NV uses only and ϵ is an error. Equation 2.7 allows us to get the optimality condition of v_0. In order to be clearer, principle results are presented in table 2.4

Distribution	$U[\mu_0-\sigma_0,\mu_0+\sigma_0]$	$N(\mu_0,\sigma_0)$
Condition for $\epsilon=0$	$\forall j, \sigma_0 \leq \frac{\mu_j-\mu_{j-1}}{2}$	$\forall j, \sigma_0 \leq \frac{\mu_j-\mu_{j-1}}{4}$
$E(\pi(Q^*))$	$E_\sigma + E_v$	$E_\sigma + E_v$
$E(\pi(Q^*))$ for linear case	equation 2.11	equation 2.11
E_v	equation 2.8	equation 2.8
E_σ	equation 2.9	equation 2.10
v_0^*	equation 4.14	equation 4.15

Table 2.4: Expected profit function for uniform and normal distributions

We have:

$$E(\pi(Q^*)) = \sum_{i=0}^{n-1} \int_0^{Q^*+\mu_0-\mu_i} (v_i - v_{i+1})(x+\mu_i-\mu_0)f(x)dx;$$

and

$$\sum_{i=0}^{n-1}(v_i - v_{i+1})F[Q^*+\mu_0-\mu_i] = v_0 - w$$

For a NVP with uniformly distributed demand for example, if $\forall j, u_j - u_{j-1} > \sigma_0/2$. There must be a i that if $j > i$, then $F(Q^*-\mu_j+\mu_0) = 0$ and if $j < i$, we have $F(Q^*-\mu_j+\mu_0) = 1$.

Thus, $F(Q^*-\mu_i+\mu_0) = \frac{v_i-w}{v_i-v_{i+1}}$.

We have $F(Q^*-\mu_i+\mu_0) \geq 0$, as a result, $\frac{v_i-w}{v_i-v_{i+1}} \geq 0$, thus $v_i \geq w$. In other words, the inventory with quantity Q^* is all sold with prices higher than the purchasing price w. In fact, when the total discount number n is fixed, the latter discounts are unused, as a result, if the NV cuts the price slower in the beginning (before the price is cutten to be lower than w), more discounts are really used. This explains why the best discount scheme cuts down the price slowly in the beginning of the selling season and faster in the ending, in our numerical examples. Though the demand distributions have higher uncertainties, it can be explained in the same way. The optimal expected profit can be written as: $E(\pi(Q^*)) = (v_0-v_1)\mu_0+(v_1-v_2)\mu_1+\cdots+(v_{i-1}-v_i)(\mu_{i-1})+\int_0^{Q^*+\mu_0-\mu_i}(v_i-v_{i+1})(x+\mu_i-\mu_0)f(x)dx$. We have $\int_0^{Q^*+\mu_0-\mu_i}(v_i-v_{i+1})(x+\mu_i-\mu_0)f(x)dx = (v_i-w)\mu_i - \frac{\sigma_0}{4}(v_i-v_{i+1})(1-(2\frac{v_i-w}{v_i-v_{i+1}}-1)^2)$. Then, $E(\pi(Q^*)) = E_v + E_\sigma$, with

$$E_\sigma = -\frac{\sigma_0}{4}(v_i-v_{i+1})(1-(2\frac{v_i-w}{v_i-v_{i+1}}-1)^2) = O(\sigma_0)$$

$$E_v = (v_0-v_1)\mu_0 + (v_1-v_2)\mu_1 + \cdots + (v_{i-1}-v_i)(\mu_{i-1}) + (v_i-w)\mu_i \qquad (2.8)$$

Lemma 2. *For an additive price dependent demand with uniform distribution($U[\mu_0-\sigma_0,\mu_0+\sigma_0]$), the optimal expected profit $E(\pi(Q^*))$ is the sum of E_v, E_σ and an error ϵ; $E_\sigma = O(\sigma_0)$, a function of the uncertainty σ_0; if $\forall i$, $\sigma_0 \le \frac{\mu_i-\mu_{i-1}}{2}$, $\epsilon = 0$.*

Similarly, for any demand distribution function who has an upper bound and a lower bound, the optimal expected profit can be developed to the sum of E_v, E_σ and ϵ. The most used distributions, like normal distribution, Poisson distribution, can be approximated by bounded distributions. For example, we can use triangular distribution to approximate normal distribution. Here we give the expressions of E_σ for normal distribution and uniform distribution and the conditions that makes $\epsilon = 0$:

For uniform distribution, if $\forall j, \sigma_0 \le \frac{\mu_j-\mu_{j-1}}{2}$, $\epsilon = 0$,

$$E_\sigma = -\frac{\sigma_0}{4}(v_i - v_{i+1})(1-(2\frac{v_i-w}{v_i-v_{i+1}}-1)^2) \tag{2.9}$$

For normal distribution, if $\forall j, \sigma_0 \le \frac{\mu_j-\mu_{j-1}}{4}$, $\epsilon = 0$,

$$E_\sigma \approx -\sigma_0^2(v_i - v_{i+1})f(F^{-1}(\frac{v_i-w}{v_i-v_{i+1}})) \tag{2.10}$$

The " $\approx$ " comes from the fact that it's not a finite distribution.

A numerical example can well verify these results(c.f. Appendix 5). It's practically feasible for the manger to approximate the expected profit by $E_\sigma + E_v$, and numerically it's faster. Taking the classical NVP with uniform distribution for example: $E_v = (v_0-v_1)\mu_0 + (v_0-w)\mu_0 = (v_0-w)\mu_0$; $E_\sigma = -\frac{\sigma_0}{4}(v_0-s)(1-(2\frac{v_0-w}{v_0-s}-1)^2)$; if $\forall i, \sigma_0 \le \frac{u_1-u_0}{2}$, $\epsilon = 0$, in fact this condition can be satisfied for all σ_0 because $u_1 = \infty$. Developing equation 2.6, we get the same equation as $E_v + E_\sigma$.

For a NV with additive demand, $x_i = \mu_i + \varepsilon$, and $\mu_i = a - bv_i$. a and b are both positive constants, and ε is a random variable with a probability density function and cumulative distribution function with a mean of zero. When discounts are linearly decreasing, $v_i = v_0 - (v_0 - s)i/n$. In conditions that make $\epsilon = 0$, the optimal expect profit is developed to:

$$E(\pi(Q^*)) = E_\sigma + (-\frac{b}{2} - \frac{b}{2n})v_0^2 + av_0 + \frac{b(w+s)}{2n}v_0 + \frac{b}{2}w^2 - aw - \frac{bws}{2n} + \frac{b}{2}(w-v_i)(v_{i+1}-w) \tag{2.11}$$

Subtract $E(\pi(Q^*))$ by E_σ and the last term, it turns to be a parabola of v_0 and hyperbola of n.

$$E(\pi(Q^*)) = E_\sigma + \frac{b}{2}(v_0 - s)^2 + (v_0 - w)(a - bv_0) - \frac{b}{2n}(v_0 - w)(v_0 - s) + \frac{b}{2}(w - v_i)(v_{i+1} - w) \tag{2.12}$$

Then

$$E(\pi(Q^*)) = E_\sigma + (-\frac{b}{2} - \frac{b}{2n})v_0^2 + av_0 + \frac{b(w+s)}{2n}v_0 + \frac{b}{2}w^2 - aw - \frac{bws}{2n} + \frac{b}{2}(w - v_i)(v_{i+1} - w) \tag{2.13}$$

In conditions that make $\epsilon = 0$, the expected profit function is the sum of E_σ with order $\frac{1}{n^2}$, the last term with order $\frac{1}{n^2}$ (we have $0 \leq \frac{b}{2}(w - v_i)(v_{i+1} - w) \leq \frac{b}{2}\frac{(v_0-s)^2}{4n^2}$) and a function of v_0 and n.

This function is a parabola of v_0 and hyperbola of n. This explains the numerical results that the expected profit increases with n but has an upper limit.

We define v_p the optimal condition of the parabola, $v_p = \frac{2na+b(w+s)}{2b(n+1)}$. Obviously, the v_p increases with n.

2.5 Optimal pricing and ordering decisions for multiplicative price-dependent demand

In the case of multiplicative price-dependent demand, [37] assume that: $X_0 = \mu(v_0)\varepsilon$, ε is independent of price. The demanded quantity till the i-th discount period can be expressed as: $X_i = \mu(v_i)\varepsilon$. After the demand in the regular period is realized, we assume that ε becomes known and takes the value ε_0. Then $x_0 = \mu(v_0)\varepsilon_0$ and $x_i = \mu(v_i)\varepsilon_0$. So we have:

$$x_i = x_0 \frac{\mu_i}{\mu_0} \tag{2.14}$$

Let us provide some argument in support of the assumption of multiplicative price-dependent demand. The actual sale of the product during the season depends on whether or not customers like that particular product. In terms of modeling, this is represented through the random term that affects sales. The higher the random terms (compared to the average value of one), the larger the actual sales. And conversely, the lower the random terms (compared to the average value of one), the lower the actual sales. If we assume that customers coming during the sales season will statistically have the same behavior as those coming during the regular season, it is therefore consistent to use the same random term to reflect whether the product under consideration is successful. Let us illustrate this further. Consider a sweater with two colors. Color 1 has been very successful during the regular season and the actual sales were 40% higher than the expected value. On the other hand customers did not like very much Color 2 and the actual sales were 30% lower than expected. It is reasonable to assume that the sales of Color 1 sweater during the sales season will be 40% higher than expected while the sales of Color 2 sweater during the sales season will be 30% lower than expected.

2.5.1 Optimal expected profit and optimal order quantity

Replace x_i in the profit function $\pi(Q)$ by x_0 (equation 2.0). The expected profit function is derived in the Appendix 3.

Lemma 3. *The expected profit function $E(\pi(Q))$ is concave.*

Proof. The proof is provided in the Appendix(c.f. Appendix 4). $\square$

The condition of optimal order quantity is given by:

$$\sum_{i=0}^{n-1}(v_i - v_{i+1})F[Q^* \frac{\mu_0}{\mu_i}] - v_0 + w = 0 \tag{2.15}$$

When n=1, we get the optimality condition for the classical NVP:

$$F(Q*) = \frac{v_0 - w}{v_0 - s}$$

Similar to the additive demand case, the optimal expected profit is:

$$E(\pi(Q^*)) = \sum_{i=0}^{n-1} \int_0^{Q^* \frac{\mu_0}{\mu_i}} (v_i - v_{i+1}) \frac{\mu_i}{\mu_0} x f(x) dx \tag{2.16}$$

Let's note that when n=1, we get the profit for the classical NVP:

$$E(\pi(Q^*)) = \int_0^{Q^*} (v_0 - s) x f(x) dx$$

2.5.2 Approximation of the expected profit function

The approximation method in the additive case inspires us to do the same thing in this multiplicative case in the same way. For this reason we propose the approximation method first and then after we will give the numerical examples for both section 5.1

and 5.2. In this way we can make comparisons between results in these two sections. The equation of the expected profit can therefore be decomposed in 3 components too:

$$E(\pi(Q^*)) = E_\sigma + E_v + \epsilon \tag{2.17}$$

E_σ is a part of expected profit depending on σ only, E_v is a part of expected profit depending on the prices the NV use only and ϵ is an error. Equation 2.17 allows us to get the optimality condition of v_0. In order to be clearer, principle results are presented in Table 2.5.

Distribution	$U[\mu_0 - \sigma_0, \mu_0 + \sigma_0]$	$N(\mu_0, \sigma_0)$
Condition that $\epsilon = 0$	$\forall j, \sigma_0 \leq \frac{\mu_j - \mu_{j-1}}{2}$	$\forall j, \sigma_0 \leq \frac{\mu_j - \mu_{j-1}}{4}$
$E(\pi(Q^*))$	$E_\sigma + E_v$	$E_\sigma + E_v$
Exponential case	equation 2.21	equation 2.21
E_v	equation 2.18	equation 2.18
E_σ	equation 2.19	equation 2.20

Table 2.5: Expected profit function for uniform and normal distributions

$$\begin{aligned} E_v &= (v_0 - v_1)\mu_0 + (v_1 - v_2)\mu_1 \\ &\quad + \cdots + (v_{i-1} - v_i)(\mu_{i-1}) + (v_i - w)\mu_i \end{aligned} \tag{2.18}$$

Lemma 4. *For a uniform distribution($U[\mu_0 - \sigma_0, \mu_0 + \sigma_0]$), the optimal expected profit is the sum of E_v, E_σ and ϵ; $E_\sigma = O(\sigma_0)$; if $\forall i$, $\sigma_0 \leq \frac{\mu_i - \mu_{i-1}}{2}$, $\epsilon = 0$.*

Similarly, for any demand distribution function who has an upper bound and a lower bound, the optimal expected profit can be developed to the sum of E_v, E_σ and ϵ. Here we give the expressions of E_σ for normal distribution and uniform distribution and the conditions that makes $\epsilon = 0$.

For uniform distribution, if $\forall j, \sigma_0 \leq \frac{\mu_j - \mu_{j-1}}{2}$, $\epsilon = 0$, $E_\sigma =$

$$-\frac{\sigma_0}{4}\frac{\mu_i}{\mu_0}(v_i - v_{i+1})(1 - (2\frac{v_i - w}{v_i - v_{i+1}} - 1)^2) \tag{2.19}$$

For normal distribution, if $\forall j, \sigma_0 \leq \frac{\mu_j - \mu_{j-1}}{4}$, $\epsilon = 0$, $E_\sigma \approx$

$$-\sigma_0^2\frac{\mu_i}{\mu_0}(v_i - v_{i+1})f(F^{-1}(\frac{v_i - w}{v_i - v_{i+1}})) \tag{2.20}$$

For multiplicative demand, $x_i = \mu_i \varepsilon$, where $\mu_i = a v_i^{-b}$. In this case, a and b are also both positive constants with the additional restriction that $b > 1$, and ε is a random variable with a probability density function and cumulative distribution function with a mean of 1. A special case is when the discounting prices are exponentially declining, we can simplify equation 2.18. $t^n = s/v_0$ and t is the ratio. We have always an i that $v_{i+1} < w \leq v_i$.

$$E(\pi(Q^*) = \frac{1 - t + v_0^{-1} t^{-bi}(v_{i+1}(1 - t^{-b}) + w(t^{1-b} - 1))}{1 - t^{1-b}} a v_0^{1-b} + E_\sigma \tag{2.21}$$

Fix v_0, when n increases, $E(\pi(Q^*)$ tends to the limit given in equation 2.22

$$\lim_{n \to \infty} E(\pi(Q^*) = a v_0^{1-b} \frac{1 - (w/v_0)^{1-b}}{1 - b} \tag{2.22}$$

Fix n, for $w = v_i$ or $w = v_{i+1}$, $E(\pi(Q^*) =$

$$a v_0^{1-b} \frac{1 - (\frac{s}{v_0})^{1/n}}{1 - (\frac{s}{v_0})^{(1-b)/n}} (1 - (\frac{w}{v_0})^{1-b}) \tag{2.23}$$

No direct expression for optimal initial price is obtained. But equation 2.23 can help a manager to get an approximate value of it. Numerical examples will show more insights on it.

2.5.3 Numerical analysis

We use normal distributed demand $N(\mu_0, \sigma_0)$ in our examples. Let's note that other distributions will also work well. Give $s = 2, w = 3, a = 4000, b = 4$. The optimal expected profit is obtained by equation 2.16.

2.5.3.1 First case: discount prices are exponentially declining

We work on the multiplicative price-dependent demand in an exponential declining discount case. According to lemma 4, $E(\pi(Q^*)) - E_\sigma - E_v = \epsilon$, and $\epsilon = 0$ in the conditions obtained. According to equation 2.22, the expected profit should have a limit close to $a v_0^{1-b} \frac{1-(w/v_0)^{1-b}}{1-b} = 38.7$. Set $v_0 = 5$, $\sigma_0 = 0$(deterministic demand), $0.1\mu_0$, $0.2\mu_0$, $0.3\mu_0$, and n increases from 2. The expected profit $E(\pi(Q^*))$ is calculated by equation 2.16; Figure 2.5 shows the values of $E(\pi(Q^*)) - E_\sigma$ and E_v.

The graph shows that $E(\pi(Q^*)) - E_\sigma$ and E_v increase with the number of discounts; the increase speed is decreasing and tends to be 0 when $n \to \infty$. These results are

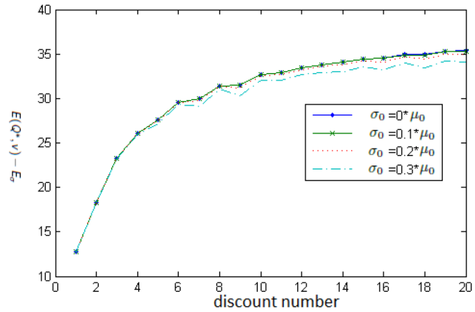


Figure 2.5: Expected profit as function of discount number n

similar to the additive demand case. When $n < 7$, $\epsilon = 0$ for these values of σ_0; then ϵ will increase with n, but even at n=20, $\epsilon < 3.6\% E_v$. Repeat computations with different combinations of s, w, a, b, v_0, we get similar results. So it is practically feasible to calculate the expected profit by the sum of E_v and E_σ. And numerically it's much quicker.

2.5.3.2 Second case: the prices are not exponentially declining

We take in our analysis $n = 6$, $\sigma_0 = 0.1\mu_0$, and v_0 changes from 3 to 12. The discount prices were produced as: $\alpha_1 v_0$, $\alpha_2 v_0$, $\alpha_3 v_0$, $\alpha_4 v_0$, $\alpha_5 v_0$, s. $\alpha_i = (\frac{s}{v_0})^{i/n}(1 + coe(n - i)i)(i = 1, ..., n)$. We change coe to control the perturbation of the exponential discount scheme. When $coe = 0$, it is the exponentially declining case. We show here 7 series of discounts (coe=-0.03, -0.02, -0.01, 0, 0.01, 0.02, 0.03), and compute the expected profit by equation2.16 (Figure 2.7). Figure 2.7 shows also the approximate expected profit value for the exponential discount scheme (equation 2.23).

As Figure 2.7 shows, the approximate curve is concave, it has the optimal expected profit(29.7) at the initial price $v_0 = 6.3$, while the equation 2.16 gives two poles (scheme 0). The first maximum (30.2, which is also the global maximum) occurs at $v_0 = 6.1$. The difference between these two initial prices is 3.3%, and 2% between the optimal expected profits. We find that the two curves coincide at $v_0 = 6.7$: in this case, $v_4 = w = 3$, this is a special case when equation 2.23 equals to equation 2.21. When $v_0 < 5$, these two curves share the same values. But error of the approximate equation 2.23 turns bigger when initial price is bigger. This error comes from our assumption: $v_i = w$, while in fact, $v_i \leq w < v_{i-1}$. This assumption gives an error between $[0, v_{i-1} - v_i)$. In this case,

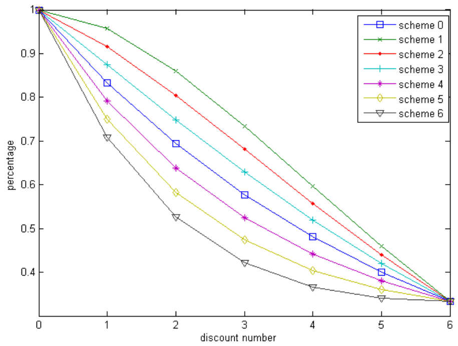


Figure 2.6: Discount percentages at $v_0 = 6$ for different schemes

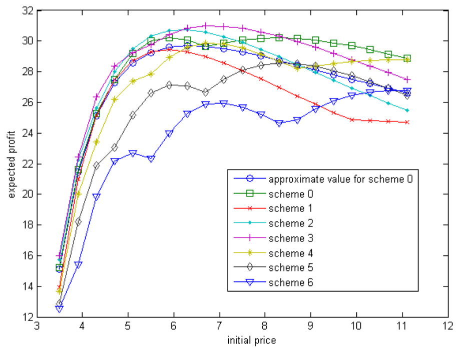


Figure 2.7: Expected profit as function of initial price

$v_{i-1} - v_i = ((\frac{s}{v_0})^{(i-1)/n} - (\frac{s}{v_0})^{i/n})v_0$, it increases with v_0.

The expected profit can have several poles (e.g.scheme 6). Comparing the exponentially declining scheme to others, we get similar results to the additive case. The discount scheme 3 with *coe* $= 0.01$ gives the maximum value (31.0) of optimal expected profit, and it's close to the exponentially declining case(30.2).

To conclude, the best discount scheme happens when the selling price is cut down a little slower than the exponential case at the beginning of the selling season; the exponentially declining discount scheme brings an optimal expected profit which is very close to the best discount scheme; when the manager choose the exponentially declining discount scheme, an approximate function can be used to get the optimal initial price.

2.6 Conclusion

In this chapter, we extend the classical NVP to the case where demand is price dependent and multiple discounts are used to sell excess inventory, which is disposed of at the end of the selling season. We determine the optimal order quantity, initial selling price and discount scheme.

We develop a general profit formulation for a NVP having multiple discounts. We prove the concavity of the expected profit for both additive and multiplicative price-dependent demand cases under general demand distributions with no limit on the discount scheme (in other words, it works for any discount scheme with decreasing percentages). We then develop the optimality conditions of the order quantity for both cases. Furthermore, we provide a simple expression of the expected profit corresponding the optimal order quantity.

Numerical examples show that the expected profit increases with the discount number, but it has an upper bound. It is not profitable to use too many discounts, because the increasing speed of the expected profit decreases and tends to zero.

For additive and multiplicative demand, a common result is that it is not good to cut down prices at a high speed in the beginning of the season. The optimal scheme in our examples cuts the price in a slow manner at the beginning of the season and faster at the end.

An approximation method is also developed. We write the profit function as the sum of a function of price, a function of uncertainty and an error term. We derive conditions where this error is zero and the optimality conditions of the initial selling price. These expected profit equations show a much clearer insight into the impact of initial price and discount number on the expected profit and confirm our numerical results. In additional case with linear discount scheme, the optimal initial price increases with discount number.

Similar to [27] and [78], our work is limited to the assumption that the additional demand related to each discount has a fixed value or is proportional to the demand realized during the regular selling period. Practically it can be different and the supplementary demand related to each discount is a random variable. An ambitious future research would be to investigate the multi-discount NVP by supposing that the supplementary demands related to each discount is a random variable.

Another point is related to the fact that our numerical examples show that the expected profit corresponding to the optimal order quantity is concave in function of the initial selling price. It would be interesting to prove it analytically. If this property is proved, the program for solving the optimal initial price can then be simplified by a Golden Section method.

Future research can address several extensions of our model. An extension considering the discounting cost will make it possible to obtain the optimal discount number. Such a cost is observed in practice (advertising cost, marking cost,etc.). The complexity of the problem will increase, so heuristic procedures may have to be used. The optimal discount scheme is not completely solved in this chapter, it will also be an interesting point for future research. Other extensions can deal with the objective of maximizing the probability for achieving a target profit or assume a second purchasing opportunity during the selling season.

3

Assortment and Demand Substitution in a Multi-Product NVP

Retail stores are confronted to make ordering decisions for a large category of products offered to end consumers. In this chapter, we consider a multi-product NVP with demand transfer (the demands of products not included in the assortment proposed in the store are partly transferred to products retained in the assortment) and demand substitution between products that are included in the assortment. We focus on the joint determination of optimal product assortment decision and optimal order quantities for products that are included in the assortment to optimize the expected total profit. Computational algorithms are presented to solve the problem. We compare five policies that can be used in practice by developing a thorough numerical study which reveals some interesting managerial implications.

3.1 Introduction

Product variety is a key element of competitive strategy. Demand for variety comes both from the taste of diversity for an individual consumer and diversity in tastes for different consumers. For instance, Coca-Cola has a product portfolio of 3,500 beverages spanning from sodas to energy drinks to soy-based drinks [83]. Many retailers become successful by offering a wide range of product assortment. Supermarkets such as Wal-

Mart and Carrefour are good examples from grocery retailing offering a range of 100,000 products in stores. However, despite the advantages of product variety, the full range of variety cannot be supplied generally, owning to the increase in inventory, shipping, and merchandise presentation (i.e. product display cost), etc. In a Carrefour supermarket, for example, only a part of coca-cola beverages among the whole product category is displayed. Within this context, the optimization of product assortment (i.e. the products that will be offered for purchase within the store), and the order quantity for each product, is a relevant decision that retailers face.

By considering multiple products, two important factors should be considered to optimize the assortment and the optimal order quantities. First, product variety brings possible substitution when underage happens: the different variants of the same product (variants are products of different colors for instance), may act as substitutes when the consumer finds that a product is out of stock. A survey reports that only 12-18% of shoppers said that they would not buy an item on a shopping trip if their favorite brand-size was not available; the rest indicated that they would be willing to buy another size of the same brand, or switch brands [84]. Second, besides the purchasing cost which increases with the order quantity, there is a fixed cost associated with each variant of product included in the assortment, e.g. the material handling and merchandise presentation cost stemming mainly from the space and labor cost required to display products in the store, etc. In these situations, the fixed cost will clearly push to reduce the assortment size (the number of products included in the assortment) and then affect the optimal order quantities.

This chapter considers a Multi-Product NVP with Demand Substitution where we aim at determining the optimal product assortment and product order quantities considering two factors that are substitution and demand transfer. We develop a model that captures the demand transfer effect when some products are unlisted (not included in the assortment). We use the Monte Carlo method to solve the multi-product NVP under substitution. The analysis of illustrative examples shows that assortment optimization and substitution may have significant effects on the expected optimal profit.

The rest of this chapter is organized as follows. Section 3.2 presents the literature related to the model we present in this chapter. In Section 3.3, we present the multi-product NVP under demand substitution and transferring. In Section 3.4, we present five decision policies to solve the joint optimization of assortment and optimal order quantities and give computational algorithms. In Section 3.5, numerical examples are provided. Section 3.6 contains some concluding remarks.

3.2 Literature review

The bulk of the literature has focused on supply chains that deal with a single product type. However, supply chains often supply many products that are variants of a common product, and that may act as substitute products. Hence, the assortment is an important decision to be defined. Therefore in this section, first we review the earlier achievements on product substitution and then we consider papers that deal with both product assortment and product substitution. All these achievements are in the area of the NVP.

The topic of substitution in inventory management first appears in [48]. Papers on this topic can be divided into 3 categories according to the substitution type: papers of the first category deal with one-direction substitution or firm-driven substitution, where only a higher grade (e.f. quality, size, etc.) product can substitute a lower grade product, when the supplier makes decisions for consumers on choosing substitutes (see, e.g., [49, 50, 51, 52, 53]). For example, the retailer provides a high quality product as a substitute for a consumer who prefers a product with lower quality but is out of stock. The second category consists of papers where arriving consumers' number follows a stochastic function and consumers make purchasing decisions under probabilistic substitution when their preferred product is out of stock (see, e.g., [54] and [55]). Here consumers come one by one and choose their substitutes within the remaining products by themselves. The third category consists of papers considering that each product can substitute for other products and the fraction that one out-of-stock product is substituted by another product is deterministic. Moreover, this category can be divided into subcategories as either the two-product (see [48, 56, 57, 58, 59]) or multi-product case (see [60, 61, 62, 63]) and centralized or competitive case. In the centralized case, only one NV manages all products, thus is interested with a global profit optimization, while in the competitive case, each NV takes care of his/her own profit considering the competition with other NVs. [61] obtain optimality conditions for both competitive and centralized versions of the single period multi-product inventory problem with substitution. [64] study a multi-product competitive NVP with shortage penalty cost and product substitution. They characterize the unique Nash equilibrium of the competitive model. An iterative algorithm is developed based on approximating the effective demand by a service-rate approximation approach.

Joint assortment planning and inventory management problems with substitution have been extensively studied. We refer the reader to [65] for a comprehensive review of the recent literature. Two major types of demand modelling are used in earlier achievements: utility maximization (see [55, 66, 67]) and exogenous demand models (see [54, 68]). [66] consider a static substitution model with multinomial logit (MNL) demand distributions assuming that consumers are rational utility maximizers. They show that in this model the optimal solution consists of the most popular product. [55] study a joint assortment and inventory planning problem with stochastic demands under dynamic substitution (assuming that a consumer's choice is made from stock on hand) and general preferences where each product type has per-unit revenue and cost, and the goal is to maximize the expected profit. Assuming that consumer sequences can be sampled, they propose a sample path gradient-based algorithm, and show that under fairly general conditions it converges to a local maximum. [67] consider a single-period joint assortment and inventory planning problem under dynamic substitution with stochastic demands, and provide complexity and algorithmic results as well as insightful structural characterizations of near-optimal solutions for important variants of the problem. [54] consider a dynamic substitution model specified by first choice probabilities and a substitution matrix. They assume that the order quantities are set to achieve a fixed service level and give two bounds of the product demand. However, the final results are based on the approximation of the demand and the numerical examples are mainly in the case of items with uniform market share (the initial market share is the same for each product). The sensitivity analysis of the profit function to the use of the bounds is not done for other market share types, while practically the market share is non-identical. [68] consider also the demand cannibalization of the standard product demand owning to retailing its customized extensions.

Our work differs from earlier research in many ways. Unlike [54] who model demand by a negative binomial process and [68] who model demand as a Poisson process, our model formulation is under a stochastic distribution and is valid for general demand distributions. We consider two phenomena in a multi-product NVP: the transfer of demand owning to the unlistment of some products, then the substitution between products included in the assortment and give the formulation of the expected total profit. The problem is solved with the objective to find the optimal assortment as well as the order quantity for each product in order to optimize the expected total profit. The first order optimality condition is derived. Furthermore, we develop heuristic solutions to solve the problem. Numerical results with different market share types are presented for the different policies considered: policy 1 considering neither substitution nor assortment, policy 2 considering only assortment, policy 3 considering only substitution, policy 4 considering sequentially assortment and substitution and policy 5 considering simultaneously assortment and substitution.

3.3 Problem modeling

We consider a set of similar products. This set is defined as a product category. Each product is associated with a market share percentage p_i, which represents its market occupancy defined in terms of units of product. Each product has a unit selling price, unit purchasing cost and in case of over-stock, the product is disposed of with a salvage value. When a product is out of stock, consumers may choose other products to substitute the product in shortage. A fixed display cost K_i is payed for each product variant included in the assortment during the season. Considering a product category N that consists of n substitutable products in the market, the NV has to determine the product assortment M which consists of m product variants over the n product variants because of a trade-off: on one hand, the higher is m, the higher will be the NV sales and therefore the profit. On the other hand, the fixed cost K_i is considered for each product variant included in the category, this parameter pushes to decrease m. The other variants remaining in the set $R = N \setminus M$ will not be offered for sale in the store.

Before the selling season, the NV decides both the products to sell in the selling season, the order quantity for each product and present the selected products in the catalog. At the beginning of the selling season, the ordered products are received and consumers get information of the product variants offered by the store. Consumers preferring other products (products in set R) either not enter the store (first kind of lost sale, with a proportion L') or enter the store to choose products offered (in set M): the demand pertaining to products not included in the assortment is partly transferred to products displayed in the store. During the season, if the product variant a consumer intends to purchase is out of stock, he makes substitutions or leaves the store without purchasing any product (second kind of lost sale, with proportion L''). The objective of the NV is to maximize the expected profit considering both assortment and substitution. Figure 3.1 shows the considered model.

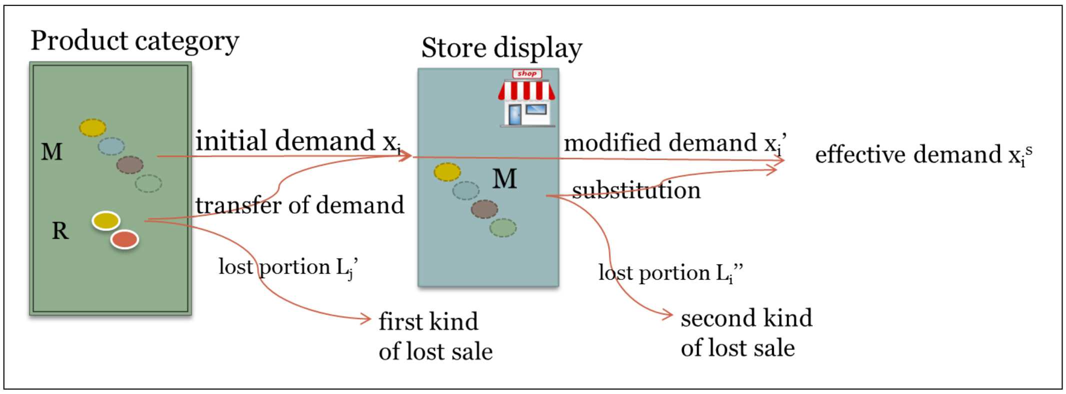


Figure 3.1: Considered model

Define the following notations used in Chapter 3:

- x a random variable representing the total demand for the entire product category. x has a continuous probability density function $f(x)$ with mean μ and standard deviation σ, and a cumulative distribution function $F(x)$,
- x_i initial demand for product i, with a probability density function $f_i(x_i)$ and cumulative distribution function $F_i(x_i)$,
- p_i the market share of demand for product i,
- $L_i^{'}$ the portion of consumers who prefer product i which is not displayed in the store and do not want to purchase another product,
- $L_i^{''}$ the portion of consumers who prefer product i which is displayed but in shortage and do not want to purchase another product,
- K_i fixed cost related to include product i in the assortment,
- v_i unit selling price for product i,
- w_i unit purchasing price cost product i,
- s_i unit salvage price for product i.

Decision variables:

- M the set of products to be included in the assortment,
- q_i the order quantity for product i, $i \in M$,
- Q the vector of order quantities, $Q = [q_i]$, $i \in M$.

The assumptions can be stated formally as follows:

ASSUMPTION 1: *The total demand distribution for the entire product category, i.e. the initial set N, is known before the selling season begins.*

ASSUMPTION 2: *Given the total demand x, the demand x_i is assumed to be equal to $p_i x$, $i \in N$.*

ASSUMPTION 3: *If consumers choose to substitute but the substitute product is out of stock, the sale is lost, i.e. there is no second substitute attempt.*

Assumptions 1, 3 is common to [54], except that we use continuous demand distributions while [54] consider binomial distribution.

3.4 Model formulation

The NV decides both which products to display within the store (the assortment) and the order quantity for each product displayed. The objective of the NV is to optimize the expected profit. We use two approaches to solve the problem: sequential optimization and global optimization. The first approach (i.e. policy 4 in Sect. 3.4.2) determines the optimal product assortment considering only the transfer of demand. Then with the obtained assortment, considering the substitutions between products displayed, we determine the optimal order quantities. In other words, the optimal assortment and order quantities are solved separately. The second approach (i.e. policy 5 in Sect. 3.4.2) is a global optimization policy considering simultaneously the transfer of demand and substitution to determine the optimal assortment and order quantities.

Besides, three other policies may be used in practice: policy 1 considers neither assortment nor substitution, policy 2 considers only assortment and policy 3 considers only substitution. Our goal in examining these policies is: 1. to understand qualitatively any distortions that might be introduced in inventory decisions if one ignores substitution effects (comparison between policy 2 and 4), 2. to gauge the impact of assortment on the expected profit (comparison between policy 1 and 2 and between policy 3 and 5), and 3. to understand any distortions that might be introduced if one considers the assortment and substitution effects independently (comparison between policy 4 and 5).

3.4.1 Modeling the transfer of demand

Additional notations:

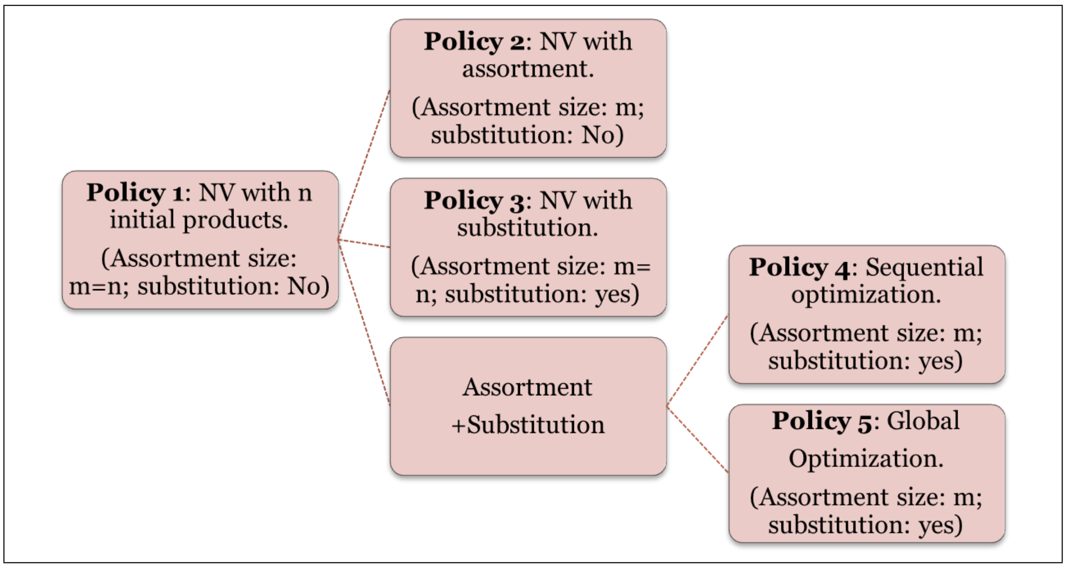


Figure 3.2: Policies analysed

$x_i^{'}$	modified demand for product i considering demand transfer effect, with a probability density function $f_i^{'}(x_i^{'})$ with mean $\mu_i^{'}$ and standard deviation $\sigma_i^{'}$, and cumulative distribution function $F_i^{'}(x_i^{'})$, $i \in M$,
x_i^s	effective demand for product i considering both demand transfer and substitution effect, $i \in M$,
$p_i^{'}$	the new market share proportion of demand for product i after the transfer of demand, $i \in M$,
α_{ij}	the fraction of consumers that purchase product j as a substitute when product i is out of stock, $i, j \in M$.

When a product variant j of the set R is unlisted, a percentage L_j of its demand is lost. The rest of the demand pertaining to product j is distributed among products of the set M. The additional demand transferred to each product i ($i \in M$) is:

$$\frac{p_i}{\sum_i p_i} \sum_{j \in R} [(1 - L_j^{'})x_j] \tag{3.1}$$

After the transfer of demand, the modified demand (the sum of initial demand and additional demand) $x_i^{'}$ for each product i ($i \in M$) is obtained as:

$$p_i x + \frac{p_i}{\sum_{i \in M} p_i} \sum_{j \in R} [(1 - L_j^{'})x_j] = p_i x(1 + \frac{\sum_{j \in R} [p_j(1 - L_j^{'})]}{\sum_{i \in M} p_i}) \tag{3.2}$$

After demand transfer, the new market share $p_i^{'}$ of product i is therefore:

$$p_i^{'} = p_i(1 + \frac{\sum_{j \in R}[p_j(1 - L_j^{'})]}{\sum_{i \in M} p_i}) \tag{3.3}$$

Property 1: The probability density function for the modified demand $x_i^{'}$ for product i denoted as $f_i^{'}(x_i^{'})$ follows:

$$f_i^{'}(x_i^{'}) = \frac{f(\frac{x}{p_i^{'}})}{p_i^{'}} \tag{3.4}$$

Proof: The proof is provided in the Appendix 1

Other properties can then be derived from Property 1.

Property 2: The cumulative distribution function for the modified demand $x_i^{'}$ for product i denoted as $F_i^{'}(x_i^{'})$ follows:

$$F_i^{'}(x_i^{'}) = F(\frac{x}{p_i^{'}}) \tag{3.5}$$

Property 3: The standard deviation of $x_i^{'}$ is :

$$\sigma_i^{'} = p_i^{'}\sigma \tag{3.6}$$

Property 4: The mean value of $x_i^{'}$ is:

$$\mu_i^{'} = p_i^{'}\mu \tag{3.7}$$

3.4.2 Modeling the various policies

The different policies of interest are presented in this section.

Policy 1: NV with n products: neither demand transfer nor substitution. In this model there is neither demand transfer nor product substitution. In fact, it can be solved as n independent classic NVP by adding a fixed cost K_i to each product i. The expected profit for product i is given by:

$$\pi(q_i) = \begin{cases} v_i x_i - w_i q_i + s_i(q_i - x_i) - K_i & \text{if } x_i < q_i \\ v_i q_i - w_i q_i - K_i & \text{otherwise} \end{cases} \tag{3.8}$$

The expected total profit is the sum of the profit for each product and is given by:

$$\begin{aligned} E(\pi(Q)) = \sum_{i=1}^{n}[\int_{0}^{q_i} (x_i(v_i - w_i) - (q_i - x_i)(w_i - s_i))f_i(x_i)dx_i \\ + \int_{q_i}^{\infty} q_i(v_i - w_i)f_i(x_i)dx_i - K_i] \end{aligned} \tag{3.9}$$

The second derivative of the expected total profit proves that it is concave with q_i, $\forall i \in N$. The optimal order quantity for product i is:

$$F_i(q_i^*) = \frac{v_i - w_i}{v_i - s_i} \tag{3.10}$$

Then the optimal expected profit is derived as:

$$E(\pi(Q^*)) = \sum_{i=1}^{n}[\int_{0}^{q_i^*} x_i(v_i - s_i)f_i(x_i)dx_i - K_i] \tag{3.11}$$

Policy 2: NV with assortment: the NV considers only the transfer of demand. For a given product set M, the demand follows a continuous probability function $f_i^{'}, \quad i \in M$ (c.f. Sect. 3.4.1). The total profit can be developed in the same way as equation 3.9 by replacing f_i by $f_i^{'}$ and n by m. The second derivative of the expected profit function proves that it is concave with q_i, $\forall i \in M$. The optimal order quantity q_i for product i respects the following equation:

$$F_i^{'}(q_i^*) = \frac{v_i - w_i}{v_i - s_i} \tag{3.12}$$

The corresponding expected profit is:

$$E(\pi(Q^*)) = \sum_{i \in M}[\int_{0}^{q_i^*} x_i(v_i - s_i)f_i^{'}(x_i)dx_i - K_i] \tag{3.13}$$

We find the same order quantity conditions as policy 1 because we consider no substitution effects in this policy. Enumeration of all possible M gives M^* that maximizes the optimal expected profit without considering the substitution effect. For some demand distributions, the expected profit equation can be simplified to a linear equation (c.f. Appendix 2).

Policy 3: NV with substitution: the NV considers only the substitution effect between n products.

The assortment is not considered. All products in N are included, i.e. the NV pays K_i for each product in N. The problem is a multi-product substitution problem similar to the one considered by [61]. During the selling season, for each product $i \in 1, \cdots, n$, a stock-out could happen and a part of unsatisfied demand will be lost with the proportion L_i. The remaining demand will be shared among the other products proportionally to their new market shares $p_j^{'}$. With a similar logic to equation 3.3, the substitution fractions α_{ij} are developed as:

$$\alpha_{ij} = \frac{p_j^{'}(1 - L_i^{''})}{\sum_{k \neq i, k \in N} p_k^{'}} = \frac{p_j(1 - L_i^{''})}{\sum_{k \neq i, k \in N} p_k} \tag{3.14}$$

The effective demand (the real functional demand after demand transfer and substitution) x_i^s for product i, which is the sum of the modified demand $x_i^{'}$ and the additional demand for product i received from other out-of-stock products caused by substitution. We have:

$$x_i^s = x_i^{'} + \sum_{j \neq i, j \in N} \alpha_{ji}(x_j - q_j)^+ \tag{3.15}$$

Here $x^+ = max(0, x)$. The expected profit function is:

$$E(\pi(Q)) = E\sum_{i=1}^{n}[(v_i - w_i)x_i^s - (v_i - w_i)(x_i^s - q_i)^+ - (w_i - s_i)(q_i - x_i^s)^+ - K_i] \tag{3.16}$$

Then the first-order necessary optimality conditions are derived from equation 3.16 as follows:

$$P(x_i < q_i^*) - P(x_i < q_i^* < x_i^s) + \sum_{j \neq i} \frac{v_j - s_j}{v_i - s_i} \alpha_{ij} P(x_i > q_i^*, x_j^s < q_j^*) = \frac{v_i - w_i}{v_i - s_i} \tag{3.17}$$

q_i^* denotes the optimal order quantity for product i in set N. P is the probability function. Let us note that equation 3.17 is the same as the one of [61] and the fixed display cost K_i does not appear in equation 3.17. Thus the fixed cost does not change the optimal inventory decision for the NV.

Policy 4: Sequential optimization: the NV considers sequentially the transfer of demand and the substitution. We use the value of M^* obtained in policy 2 then consider the substitution effect to get the optimal order quantities and the expected total profit.

First step: the Assortment Decision: We obtain the optimal set M^* by considering only the transfer of demand. Order quantity conditions (equation 3.12) and the optimal expected profit are given by equations 3.12 and 3.13. Enumeration of all possible M gives M^* that maximizes the optimal expected profit without considering the substitution effect.

Second step: Consideration of the substitution: Once the NV determines M^*, the demand follows a continuous probability functions $f_i^{'}, \quad i \in M^*$. The substitution fractions, the effective demand and the total profit can be developed in the same way as equations 3.14, 3.15, 3.16 and 3.17 by replacing f_i by $f_i^{'}$ and n by m^* (the assortment size of M^*).

The substitution fractions α_{ij} are developed as:

$$\alpha_{ij} = \frac{p_j^{'}(1 - L_i^{''})}{\sum_{k \neq i, k \in M^*} p_k^{'}} = \frac{p_j(1 - L_i^{''})}{\sum_{k \neq i, k \in M^*} p_k} \tag{3.18}$$

The effective demand:

$$x_i^s = x_i + \sum_{j \neq i, j \in M^*} \alpha_{ji}(x_j - q_j)^+ \tag{3.19}$$

The expected profit function is:

$$E(\pi(Q)) = E\sum_{i=1}^{m^*}[(v_i - w_i)x_i^s - (v_i - w_i)(x_i^s - q_i)^+ - (w_i - s_i)(q_i - x_i^s)^+ - K_i] \tag{3.20}$$

The first-order necessary optimality conditions are derived from equation 3.20 as follows:

$$P(x_i < q_i^*) - P(x_i < q_i^* < x_i^s) + \sum_{j \neq i, j \in M^*} \frac{v_j - s_j}{v_i - s_i}\alpha_{ij}P(x_i > q_i^*, x_j^s < q_j^*) = \frac{v_i - w_i}{v_i - s_i} \tag{3.21}$$

q_i^* denotes the optimal order quantity for product i in set M^*. Let us note that the second and third term on the left-hand side of equation 3.21 equal to zero for the special case where no substitution is considered, then equation 3.21 becomes the order quantity optimality condition for the classical NVP (equation 3.12).

Policy 5: Global optimization: the NVP considers simultaneously the demand transfer and substitution effects.

To obtain the optimal set M^* determined by the sequential optimization policy (policy 4), we need to consider simultaneously the demand transfer and substitution effects.

Given a product set M^*, the modified demand $x^{'}$ and the effective demand x_i^s are derived in equations 3.4 and 3.19. The expected profit and the optimal order quantities are given by equations 3.20 and 3.21. The difference is that the set M^* is no longer given by a previous assortment decision, but has to be optimized. There are 2^n possibilities for M, we enumerate all of them and we can find M^* that maximizes the expected total profit.

3.4.3 Algorithm for policy 3, 4 and 5

Caused by the complexity of equation 3.17, one cannot obtain directly the optimal order quantities within feasible run time. Thus we use a Random-walk Monte Carlo method to find the solution. The procedure is as follows:

Step 1: Initialize Q with the values obtained by the optimal order quantity condition of M independent classic NVP; initialize the walk length λ and its limit: ϵ.

Step 2: generate n random points around Q with a distance λ to Q. And get the best point $Q^{'}$ among these n points;

Step 3: if $Q^{'}$ is better than Q, assign the value of $Q^{'}$ to Q, go to step 2. If not, halve the value of λ, if $\lambda > \epsilon$, go to step 2, otherwise, go to step 4;

Step 4: if $Q^{'}$ satisfies the optimality condition, stop, otherwise go to step 1.

In order to determine in step 3 that $Q^{'}$ is better than Q or not, we define an objective function as the difference, denoted a, between the left side and right side of equation 3.17. If the order quantities are optimal, equation 3.17 should be satisfied, thus the objective equation should be equal to zero. But in fact, zero can not be strictly realized in computation. We consider that $Q^{'}$ is better than Q if $h(Q^{'}) < h(Q)$ and the optimality condition is satisfied when $h(Q^{'}) < 0.1$ (see Figure 3.3);

Step 4 is required because when we are generating N points around Q, it is possible that they are concentrated, thus the optimality condition can not be satisfied at the end of only one random walk. The fourth step ensure the optimality condition is satisfied and ends the loop.

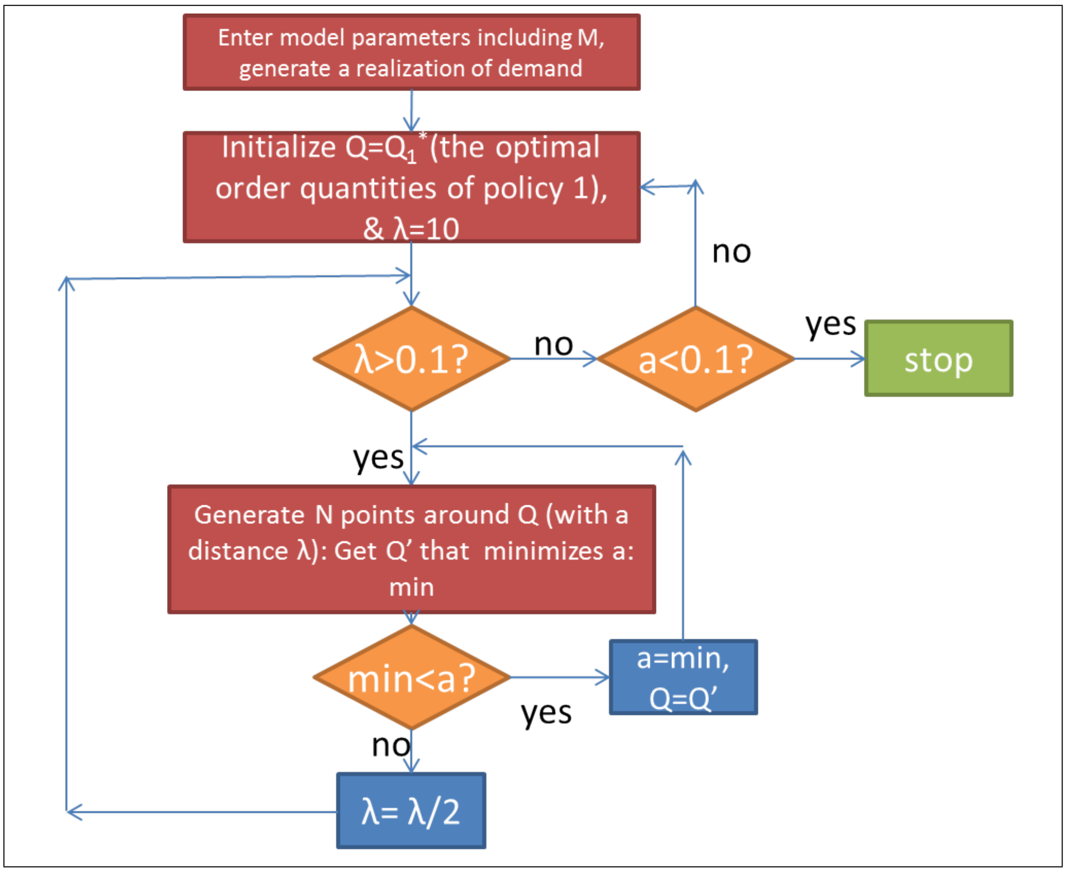


Figure 3.3: Flow chart of the algorithm for calculating the optimal order quantities

For this reason, another method is not recommended: regarding $Q^{'}$ is better than Q if $Q^{'}$ brings a better expected profit than Q, with the value of expected profit is obtained from equation 3.16. This method can fall into local maximums.

3.5 Numerical analysis

3.5.1 Numerical examples

In this section, we use a normally distributed demand, other demand distributions will also work. We consider a category of $n = 6$ initial products with total mean demand $\mu = 100$ and varying σ values. The selling price, purchasing cost, salvage value, fixed cost and lost sale portion are assumed to be the same for all products: $v = 11, w = 8, s = 3, K_i = K$, and $L_i^{'} = L_i^{''} = L$. Three market share types are considered: the linear type with p_i=(0.09, 0.12, 0.15, 0.18, 0.21, 0.25), the exponential type with p_i=(0.03, 0.06, 0.09, 0.15, 0.25, 0.42) and the uniform type with p_i=(0.17, 0.17, 0.17, 0.17, 0.17, 0.17). These simplifications facilitate the comparison between different policies and makes it easier to analyze how different performances vary with market shares. For policy 3-5, we use the Monte Carlo method to compute the optimal order quantities and expected profits. In this example, we use the default random generator in Matlab generating 10000 samples to represent the demand with normal distribution (at about a confidence level of 98% with a sampling relative error 2.3%).

By setting $K = 10$, $L = 0.3$ and exponential market shares, the expected total profit comparison for the five policies as a function of σ is given in Figure 3.4. Results show that the optimal expected profit decreases with σ and the global optimization policy outperforms the other policies and the sequential optimization does very well particularly, achieving 100%, 100%, 98.9%, 97.6% of the profit generated by the global optimization policy, respectively, for $\sigma = 10, 20, 30$ and 40.

Another result is that the substituted NV (policy 3) performs poorer than the assorted NV (policy 2) when $\sigma = 10$, but performs better as σ increases. This is because when demand uncertainty is bigger, the risk of inventory shortage or overage is more important. In this case, the substitution has a more important effect.

Optimal order quantities for each of the 6 products obtained by different policies (with $\sigma = 30$ and exponential market shares) are shown in Figure 3.5. We find that the order quantity increases with the market share value for each policy and when the assortment is considered, only high demand products are included in the assortment. As a result, the number of enumeration is largely reduced: from 2^6 to 6. Therefore, the combination possibility for a product category with n product variants is only n, which reduces significantly the computing time. Similar results are found with other values of σ.

As shown in Table 3.1, the assortment size is intensively reduced compared with Policy 1 when using the sequential optimization or global optimization policy, which indicate that the performance of the classical NV model without assortment nor substitution can be quite limited in practice.

Comparing policy 1 with policy 2 and 3: For a fixed value of σ, the optimal order quantity for each product obtained by policy 1 do not change with K or L because policy 1 ignores both the effect of demand transfer and substitution.

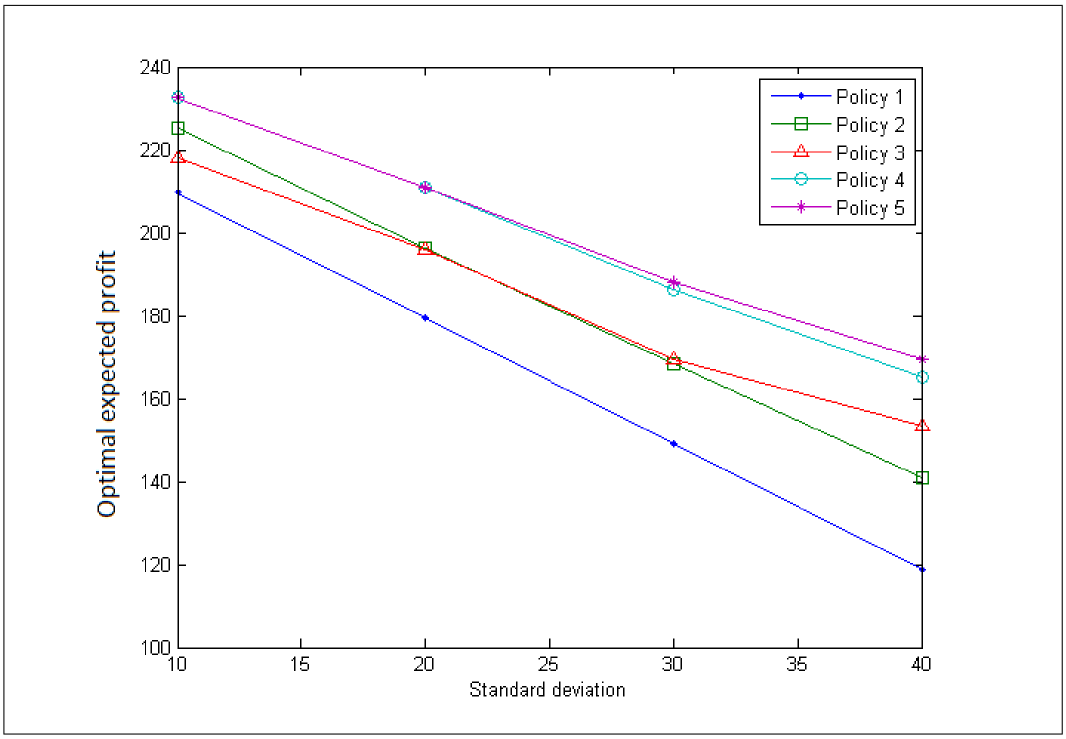


Figure 3.4: Expected optimal profit for different policies with σ=10, 20, 25, 30, 40 and K=10, L=0.3

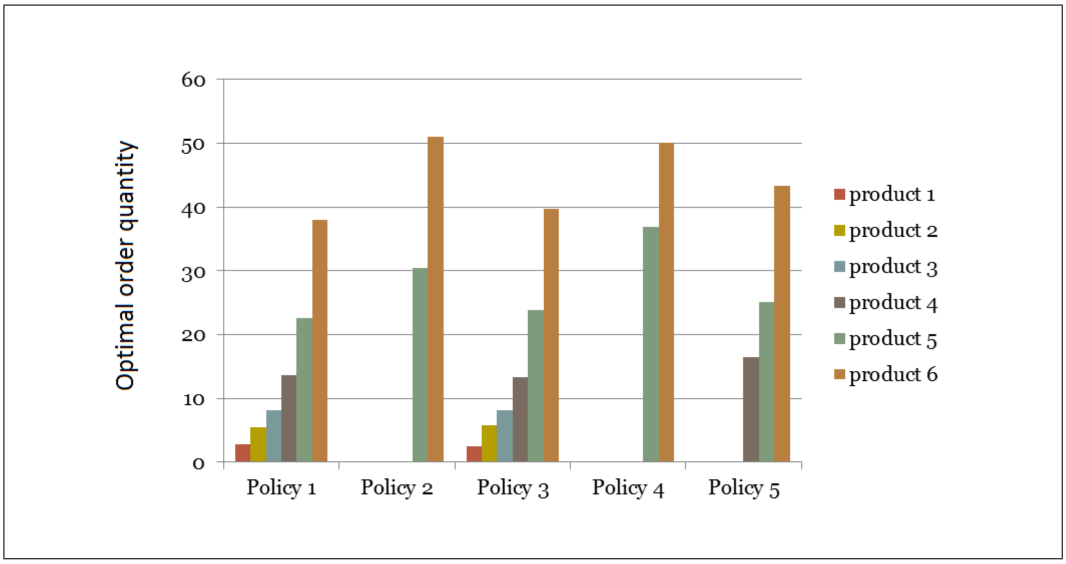


Figure 3.5: Optimal order quantities for σ=30, K=10, L=0.3

	$\sigma = 10$	$\sigma = 20$	$\sigma = 30$	$\sigma = 40$
Policy 1	6	6	6	6
Policy 2	3	3	2	2
Policy 3	6	6	6	6
Policy 4	3	3	2	2
Policy 5	3	3	3	3

Table 3.1: Optimal assortment size for different policies with $\sigma = 10, 20, 30, 40$

Considering the assortment or substitution, both increase the profit. As the assortment size decreases from n to M^*, the total fixed cost decreases, thus the profit could increase. Considering the assortment, some products are unlisted in some cases and the unlistment begins with the product having the smallest market share: firstly, the display cost K_i leads to unlist the low demand products because the revenue of these products are relatively smaller. Secondly, popular products make more sales thus bring higher profit. They have larger mean demand, thus more demand will be lost if they are unlisted, while unlisting less popular products will lose less demand. The substitution improves the profit in two aspects: on one hand, the underage cost for a product is lower because the unsatisfied demand for one product may be substituted by another product; on the other hand, the overage cost for a product is lower too, because it receives some additional substitute demand from other products.

Comparing policy 5 with policy 2 and 3: The global optimization policy leads to a higher profit compared with these two policies. The combination of assortment and substitution significantly improves the profit because the fixed cost related to including all products in the assortment can be high and the substitution brings additional demands.

Comparing policy 5 with policy 4: In our examples, policy 5 needs a computation about 10 times longer than policy 4. It obtains the same results as the sequential optimization policy when σ=10 and 20. But as demand uncertainty becomes bigger, i.e. $\sigma = 30$ and 40, the assortment size is bigger than the one obtained by policy 4, the order quantities for the products are also different, and the profit is up to 5% bigger than the one of policy 4 (see Appendix 3 to find combinations of (K, L, σ) that maximize the difference between policy 4 and 5). We try different combinations of parameters (K, L, σ) and get the same results: the assortment size and the expected total profit are not smaller than the ones obtained by sequential optimization policy.

The substitution makes it possible to enlarge the assortment size, because while one product is not profitable in policy 2, the substitution can make it receive some additional substitute demand from other products, thus this product can be profitable and is not unlisted.

3.5.2 Sensitivity to demand uncertainty

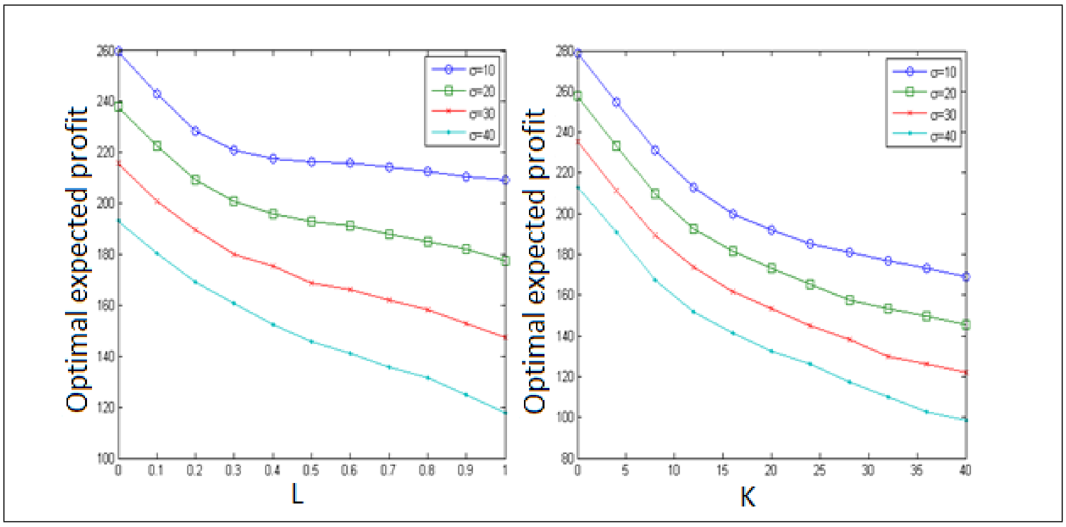


Figure 3.6: Optimal expected profit for policy 5 with the exponential market share type as a function of K (with L=0.3) or L (with K=10), with σ=10, 20, 30, 40.

A common result found in our numerical examples is that the expected profit decreases with σ, as shown in Figure 3.6. We get the same result for all values of K, L and for all three types of market shares.

For the global optimization (policy 5), the assortment size, as shown in Figure 3.7, does not respect a simple and obvious rule as σ changes.

When we fix K and change L values, for $K = 10$, the assortment size decreases with σ, except of the case L=0. Intuitions to this result are the following: when L and σ are both small, e.g. $L = 0$, $\sigma = 10$, the demand substitution benefit is less than the fixed display cost K of an additional product. For the special case where $L = 0$, σ=10, the assortment size is 1, this means there is no alternative product to buy when

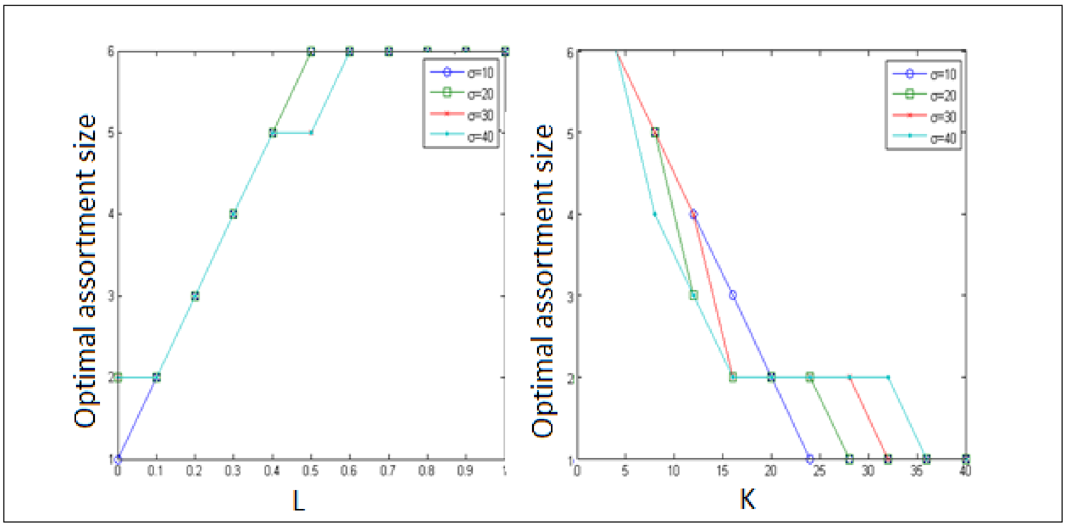


Figure 3.7: Optimal assortment size for policy 5 with the exponential market share type as a function of K (with L=0.3) or L (with K=10), with σ=10, 20, 30, 40.

the product is in shortage, thus there is no substitution. For other values of L greater than 0, the assortment size is bigger than one. As explained in Section 3, there are two kinds of lost demand: when a product variant is not included in the assortment and when a product variant is in shortage during the season. For a fixed L, on one hand, a larger assortment size means more product variants are included, thus less demand is lost, i.e. the first kind of lost demand is reduced (this increases the profit), however, there will be more display cost (this reduces the profit); on the other hand, the second kind of lost sale does not change with the assortment size because the proportion of the second kind lost demand is fixed: L. So it is a trade-off to determine the assortment size between reducing the first kind of lost sale and increasing the display cost. When σ is small, the profit coming from reducing the lost sale is bigger, so the trade-off pushes to bigger assortment size.

Then we fix L and change K values. For $L = 0.3$, the assortment size decreases with σ when $K < 20$ in our examples, and increases with σ when K is bigger. Special case is that when $K > 20$, the assortment size tends to be 1, thus there is no substitutions. In this situation, the fixed cost is bigger than the demand substitution benefit. For other cases, we have the same results and same interpretations as in the previous paragraph.

3.5.3 Sensitivity to L

Let K=10, $\sigma = 20$. Considering three market share types, i.e. linear, exponential and uniform market share, the profit and assortment size are calculated for the five policies. It is intuitive that the expected profit for policy 1 does not change with L and is always not bigger than the other policies. For other policies (policy 2,3,4,5), the profit decreases as L increases. This is because the lost sale related to demand transfer and underage substitution both increase with L. We get the following insights (See Figure 3.8):

1. As L approaches 1, the expected profit (policy 3, 4, 5) becomes identical to the one of policy 1. When $L = 1$, the substitution effect becomes zero, and demand transfer effect becomes zero too, thus equals the one of policy 1.

2. The assortment size (policy 2, 4, 5) increases with L. When L is bigger, there is more lost sale, and as explained before, the lost sale can be reduced by increasing the assortment size.

3. The expected profit of policy 2 is bigger than policy 3 when L has a small value, but becomes smaller when L increases. The reason for this is that when L is small, the assortment size for policy 2 is small, thus the NV reduces a large part of the cost by reducing the assortment size. When L is bigger, the assortment size gets bigger, the total display cost increases and the cost of policy 2 increases. As a result, the effect of considering the assortment decreases.

We have also done some numerical analysis where the two lost sale proportions are different: $L' \neq L''$. Similar properties are obtained. A special case where $L' = 0$, thus no demand transfer, is shown in Figure 3.9.

3.5.4 Sensitivity to K

Let L=0.3, $\sigma = 20$. Considering three market share types, the profit and assortment size are calculated for the five policies. It is obvious that the expected profit for policy 1 decreases linearly with K and is always not bigger than the other policies. For other policies (policy 2, 3, 4, 5), the expected profit decreases with the fixed cost. We get the following insights (See Figure 3.10):

1. As K approaches 0, the expected profit (policy 4,5) is identical to the one of policy 3. When K=0, it is always optimal to include all items in the assortment (policy

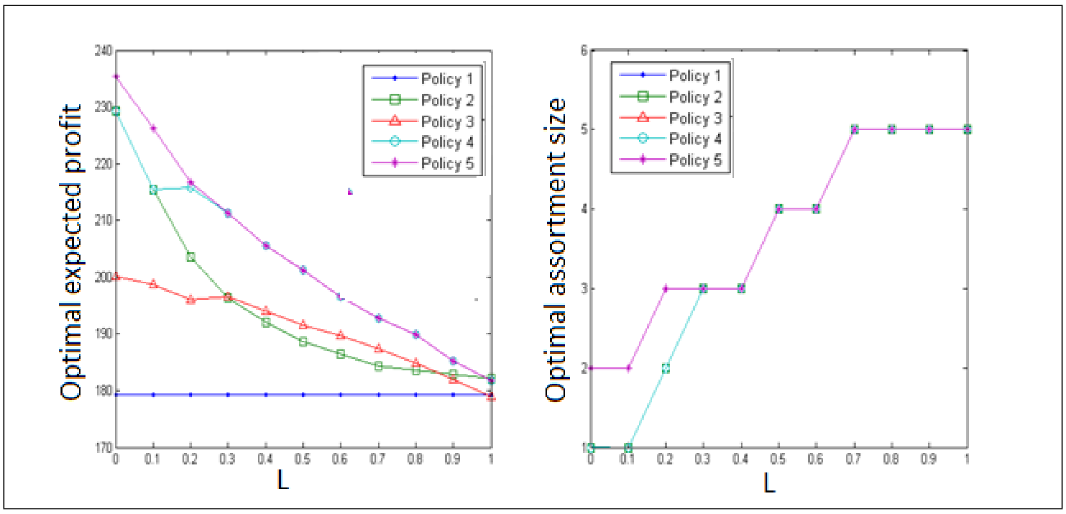


Figure 3.8: Optimal assortment size and expected profit as functions of L, for $\sigma = 20$, exponential market sharing

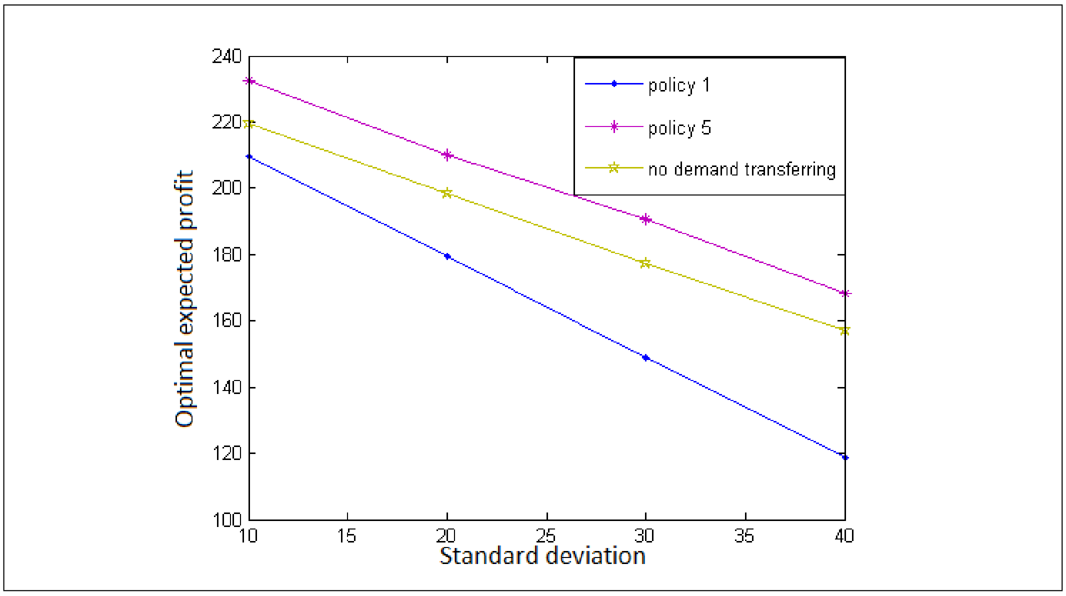


Figure 3.9: The optimal expected profit as a function of σ, with K=10, $L^{"} = 0.3$ for exponential market sharing

4,5), the assortment size is 6, thus the expected profit is the same of policy 3.

2. The assortment size (policy 2, 4, 5) decreases with K. The effect of fixed cost is more important when K is bigger. Thus for a bigger K, the assortment size is reduced.

3. The expected profit of policy 3 is bigger than the one of policy 2 when K has

a small value, but becomes smaller when K increases. The reason is when K is small, the assortment size for policy 2 is 6, thus the NV reduces a limited part of the cost by using policy 3. When K is bigger, the assortment is smaller, the NV can reduce a large part of the cost by including more products in the assortment. As a result, the effect of considering the assortment increases.

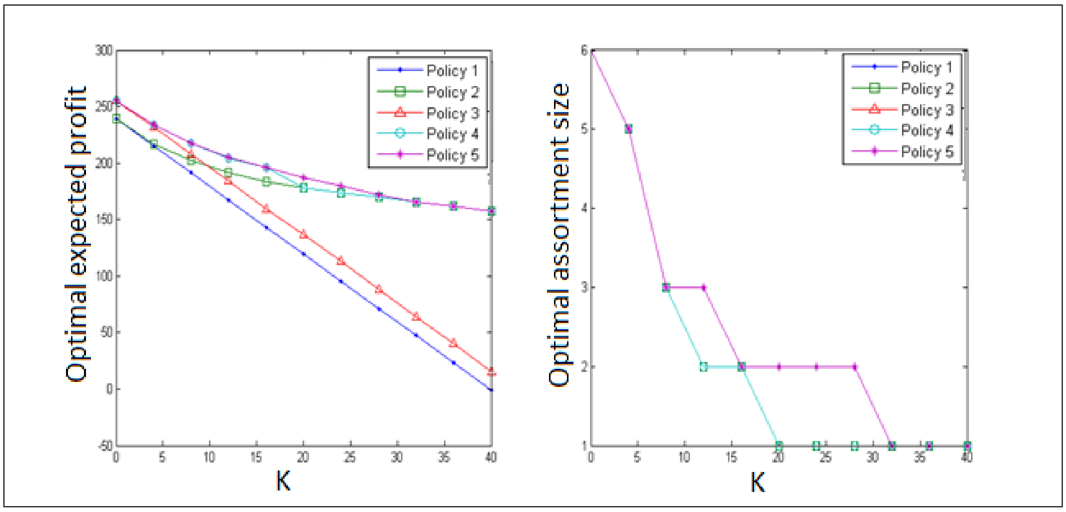


Figure 3.10: Optimal assortment size and expected profit as functions of K, for $\sigma = 20$, exponential market sharing

3.5.5 Impact of the market share type

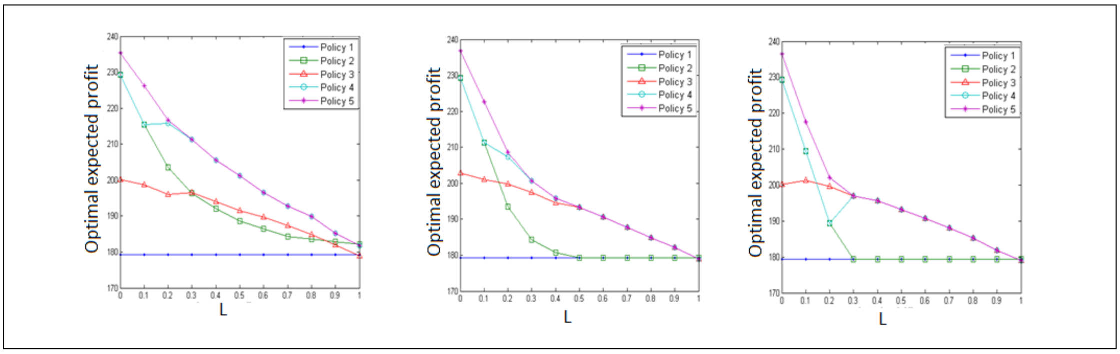


Figure 3.11: Optimal expected profit as a function of L, with $\sigma = 20$, for exponential market sharing, linear market sharing and uniform market sharing

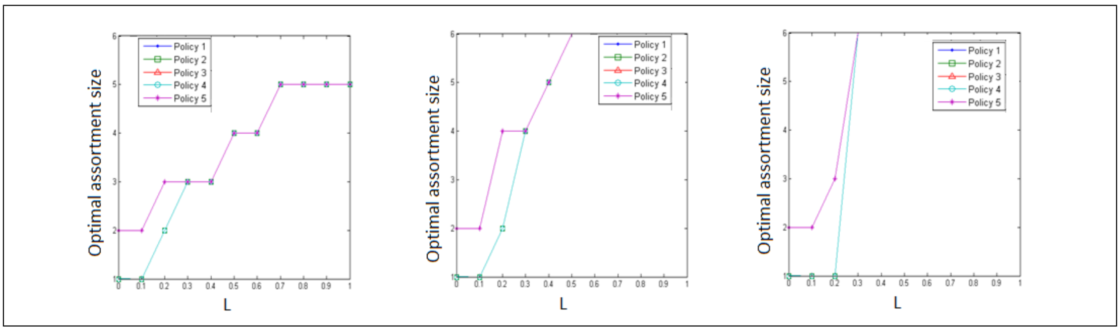


Figure 3.12: Optimal assortment size as a function of L, with $\sigma = 20$, for exponential market sharing, linear market sharing and uniform market sharing

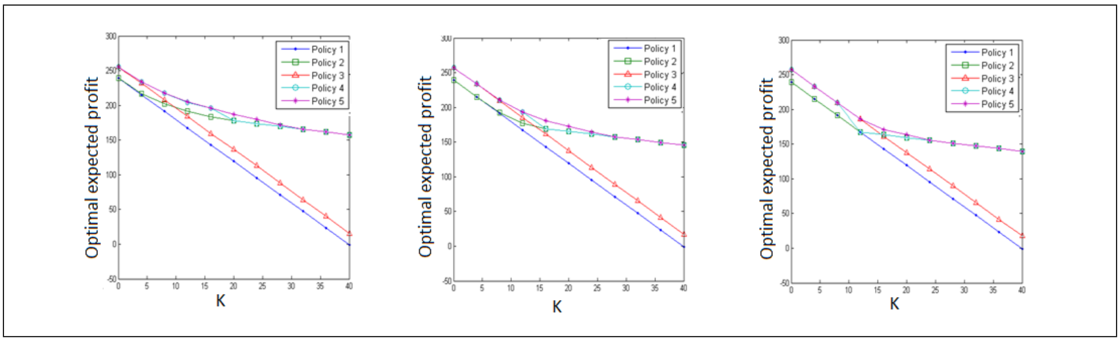


Figure 3.13: The expected profit as a function of K, with $\sigma = 20$, for exponential market sharing, linear market sharing and uniform market sharing

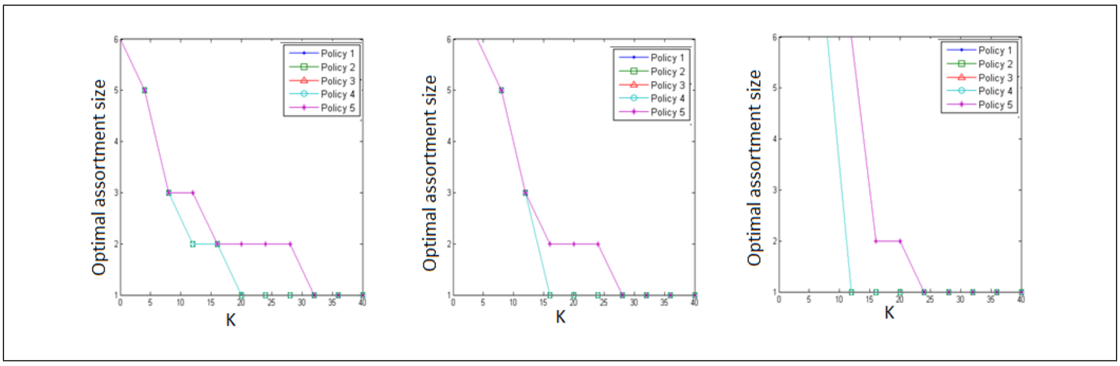


Figure 3.14: The assortment size as a function of K, with $\sigma = 20$, for exponential market sharing, linear market sharing and uniform market sharing

As shown in Figure 3.11, 3.12, 3.13 and 3.14, the type of market share has an important effect on the assortment size and the expected profit.

From the exponential market share to the uniform one, the assortment size increases faster with L. The reason is that when the market share becomes more balanced, the substitution effect is even more important. In the exponential case, the optimal assortment size cannot even reach the value of 6, even though all demand for product 1 will be lost when $L = 1$, because the profit of the first product is less than the fixed cost K.

As shown in Figure 3.14, the first product is unlisted faster with an exponential market share. It is because the first product is less profitable in this case, and will be quickly unlisted because the display cost will be larger than its profit.

Another insight is on the value of K and L for which policy 2 and policy 3 have the same expected profit. We can see that this value of L decreases when the market share becomes more balanced and the value of K increases. The balance of market share reinforce the effect of substitution and reduces the impact of fixed display cost.

3.6 Conclusion

This chapter extends the classical NVP to solve the joint optimization of product assortment and order quantities by considering demand transfer and substitution effects. We formulate the transfer and substitution fractions. A random-walk Monte Carlo method provides an efficient computational approach to get the value of the expected optimal profit and optimal order quantities for a product assortment.

Our numerical examples show new insights regarding the performances of the NVP. In particular, demand transfer and substitution have significant effects on the assortment size, expected profit, and optimal order quantities. Additionally, the sequential optimization policy and global optimization policy both bring better profit performance than considering only one effect. Sequential optimization policy shows close results to global optimization policy and the computing time is reduced up to about 10%. But in some cases the expected profit of sequential optimization policy is up to 5% less than the one of global optimization policy, thus it is necessary to use the global optimization policy to obtain the best profit. The difference between policy 1 and policy 5 increases with the value of fixed cost and decreases with the value of lost sale proportion.

With the global optimization policy, several insights can be derived from numerical results: The expected profit decreases with the fixed cost value, the fraction of lost sale and demand uncertainty. Assortment size increases with the fraction of lost sale but decrease with the fixed cost value. The total order quantity does not respect strict behaviors but it shows that the order quantity reaches its maximum when lost sale fraction is zero or 100%, and tends to decrease with fixed cost value.

The model can easily be adapted to problems with other kinds of substitution such as one-item substitution, which can be treated in the same way as our model by only changing the demand transfer and substitution equations. This could be interesting because different kinds of substitution happens for different kinds of products: in the textile industry for example, consumers could substitute to a shirt with a bigger size but probably not in the contrary way. In this case, it is a one-direction substitution.

An interesting direction, related to this model, lies in investigating the difference between two demand lose portions. As we explained in our modeling assumptions, the lost portion related to a product not displayed is expected to be larger than the one of a displayed but under-stocked product. Numerical analysis can show the impacts by examining the change of both the optimal assortment and order quantities when the NV increases the not-displayed portion.

Our work is limited by supposing that the demands of product variants are all related to the total demand, while practice, it may be not the case. Future research can be developed to a case where the demand for each product variant is independent of others' and individual demands are given. In this case, the demand transfer formulation will be different: it will be difficult to derive the distribution functions of demands for the products after demand transfer (the only case there we have found a solution is when demands are all normally distributed). However, using the Monte Carlo method, the complexity of programming for numerical results will not be increased compared with our model.

In our numerical examples, the expected profit appears to be unimodal in the order quantity of each product variant. But analytically we have not succeeded to prove it. We have actually demonstrated the non-concavity of the expected profit on each demand, but the non-unimodality is to be proven analytically. Demonstrating analytically the unimodality would enable us to cut down the programming time for numerical examples.

4

The NVP with Drop-shipping Option and Resalable Returns

As e-commerce expands, more and more products are offered online to attract internet consumers' interest. These products are often provided at consumers' home by a drop-shipper. Indeed, in recent years, drop-shipping seems to be a good option to sell products in addition to physical stores. In addition, both types of products, either sold in store or on Internet can be returned by consumers, with often a higher return ratio for those purchased on Internet. To model these two sales channel and interactions between them, we consider a NV managing both a physical store inventory and a sale channel on internet that is fulfilled by a drop-shipping option. In addition to these two supply options, we consider the possibility of reselling products that are returned by consumers during the selling season. The concavity of the expected profit is proven and the optimality condition is obtained. Various results are obtained from a numerical analysis. In particular, the expected can be 14.4% less than the optimal expected profit if the return effect is ignored. Using drop-shipping option can reduce the optimal store inventory by 31.2% and if the NV has no drop-shipping option, the expected profit can be 9.0% less.

4.1 Introduction

E-commerce is constantly growing in various industrial sectors. According to Remarkety [85], in 2015, 57.4% of the US population and 80% of the population of Japan shop online. Hence, more and more suppliers and retailers have presence on the internet to offer products which are also sold in physical stores, in order to provide end consumers a larger choice regarding the channel along which they can buy products without increasing operation costs. In apparel industry, Zara for example, uses a distribution center to provide products for physical stores as well as internet sales at the same time [86].

E-commerce thus brings a new opportunity for retailers to supply products to consumers through electric markets. Indeed, drop shipping is a recent order fulfilment approach where the retailer does not keep goods to be sold in store but instead, displays products on his/her company website, collects and transfers consumer orders to the wholesaler or the supplier, who is then in charge of shipping goods directly to end consumers.

Drop shipping can be attractive for the retailer since it does not require him/her to bear the cost of holding inventory in the store. As a result, products can be offered to the consumer at a lower unit selling price on Internet, in comparison to the unit selling price that the consumer would have to pay if the product is bought in the physical store. Drop-shipping can also be attractive for the the wholesaler/supplier by enabling him/her to sale products on the retailers' websites.

Drop shipping can be especially interesting for seasonal products. Such products have generally a short selling season and a long replenishment lead time where the order is generally placed to a distant supplier before the selling season. The NV Problem is a classical model used for such products, it aims at finding the optimal order quantity which maximizes the expected profit under probabilistic demand [8, 9]. The demand for the product is unknown before the selling season, thus the order quantity for the product should be optimized from the trade-off between two situations: if the order quantity is too large, overstock happens; if the order quantity is not enough, underage happens and lost sale causes lost profit. If the order is smaller than the realized demand, it is not possible to place another order during the season to the distant supplier. In such a case, drop shipping (i.e. ordering products from a wholesaler/supplier which is geographically closer to the retailer) can be used to fulfill demand.

One of the major issues related to e-commerce operations concerns product returns since products sold through e-commerce tend to have a higher return rate than those sold within stores [70]. This return rate can be as high as 75% for Internet sales

[87]. Hence, in many businesses such as textile or electronics, consumers have the legal right to return a product purchased online within a certain time frame if it is in good condition. Such products return to the retailer store during the selling season and can be reused as new products after some treatment by the retailer, e.g. quality examination, product repairing, re-labelling/packaging, etc. Thus it is important to consider this potential return flow when making inventory decisions.

This chapter considers a NV managing both a physical store and sales on internet fulfilled by a drop-shipping option. We also assume that returns are resalable during the selling season after a certain treatment. The objective is to optimize the order quantity (thus the store inventory that will be available at the beginning of the season) for the order placed before the selling season. As the classical NV problem, store demand (the demand of consumers shopping physically in the store) is satisfied by store inventory. The NV can also use the store inventory to satisfy internet demand and has in addition a drop shipping option for excess internet demand (i.e. a mixed fulfillment strategy is used). In case that store demand is not totally satisfied, a part of the unsatisfied store demand is substituted to Internet demand. When products are delivered, some consumers are unsatisfied and a portion of products is returned to the store. The return rates are assumed different depending on where products are supplied from (store or drop shipper) and whom products are sold to (store consumer or Internet consumer). Under these assumptions, we express the expected profit formulation and demonstrate the concavity of the function. Optimality condition is also given. The optimal expected profit equation is then derived. We present two model variants depending on whether Internet returns can be used for store demand. Some special cases are discussed. A numerical analysis is conducted leading to interesting results. We illustrate the impact of return, drop-shipping and different parameters e.g. the substitution fraction.

The rest of this chapter is organized as follows. Section 4.2 presents the related literature. In Section 4.3, we present the NV Problem with a mixed supply strategy considering product returns. In Section 4.4, we formulate the optimal drop-shipping order quantity in each case for two variants of model. The expected profit is formulated and the optimal order quantity for store inventory is developed. In Section 4.5, numerical examples are provided. Section 4.6 contains some concluding remarks.

4.2 Literature review

As e-commerce is expanding, research on drop-shipping and product returns has been increasing. Thus we review earlier achievements regarding two streams of NV Problem which are associated to our work: (1) the NV Problem with drop-shipping option and (2) the NV Problem with product returns.

[18] first solved a NVP with an emergency supply option in case of shortage. Unsatisfied demand can be satisfied by an emergency supply option. which will be analogous to the drop shipping option. [19] explicitly incorporated the drop-shipping as an emergency option into the single-period model framework and showed that it can lead to a significant increase in expected profit. [88] analyzed drop shipping for a multi-actor problem. The analysis was conducted under different power structures and included marketing and operational costs. The retailer carries out the marketing and advertising activities and the wholesalers handles the fulfillment process. [20] assessed three different organizational forms that can be used when a store-based sales network coexists with a web site order network. The three organizational forms are store-picking, dedicated warehouse-picking and drop shipping. Authors used a NV type order policy to compare the efficiency of three different models and to analyze the impact of transport costs, Internet market size and demand hazards on the profits of the stakeholders on inventory policies in the supply chain. [89] proposed that growth in product popularity leads to an increased reliance on store inventory. As [90] reported, the drop-shipping mode results in cost savings but reduces the unit profit margin, whereas the traditional mode (purchasing from the supplier with a lower unit purchasing cost and selling to consumers in the store with a higher price) provides a higher profit from each unit. [21] proposed a mixed mode that utilizes both traditional and drop-shipping modes for seasonal fashion and textiles chains, in order to take full advantage of demand fluctuation and improve the profit-making ability.

In the literature, consumer returns are typically assumed to be a proportion of products sold (e.g.[69, 70, 71, 72, 73]), which obviously implies that if more items are sold, more products will be returned from consumers. [74] empirically showed that the amount of returned products has a strong linear relationship with the amount of products sold. Based on the assumption that a fixed percentage of sold products will be returned and that products can be resold at most once in a single period, [70] investigated optimization of order quantities for a NV-style problem in which the retail price is exogenous. [75] considered a manufacturer and a retailer supply chain in which the retailer faces consumer returns. [76] also assumed that a portion of sold products would be returned and discussed the coordination issue of a one manufacturer and one retailer's supply chain. [73] examined the pricing strategy in a competitive environment with product returns. [77] considered consumer return for retailer who is confronted with two kinds of demand: one needs immediate delivery after placing an order and the other accept delayed shipment. A NV model with resalable returns and an additional order is developed. However, the model was under assumption that the total demand distribution is given and each kind of demand presents a proportion of the total demand, in addition, the concavity is not proved.

To the best of our knowledge, no research has treated the product returns issue within a mixed fulfillment strategy using both drop-shipping and store inventory. In this chapter, we model a retailer who faces product returns (such returns are not considered by [19]) from both store and Internet consumers. Earlier works ([69, 70, 71, 72, 73]), consider only the store sale channel and not both channels. Compared to the latest work that considers a comparable problem to us [77], who provided a numerical analysis based on a necessary condition without proving the concavity, we demonstrate the concavity of the expected profit function and derive the optimal order quantity condition considering independent demands for store and internet sales (i.e. two random variables instead of a unique one in [77]), different return rates instead of an identical return rate in [77], different selling prices instead of an identical selling price in [77]. In addition, we consider the effect of demand substitution in case of under-stock in store.

4.3 Problem modeling

We consider a NV which uses a combination of store inventory and drop-shipped products for fulfilling two types of demand: demand that occurs in the store and demand related to internet sales. More precisely, before the season begins, the NV orders a quantity of products Q_1, at a unit product purchase cost w_1, from the traditional (distant) supplier. During the season, those products, stored in the store, can be used to satisfy both store demand x_1 and Internet demand x_2. x_1 and x_2 are assumed to be two independent random variables. In case x_2 is not satisfied by Q_1, there is an alternative drop shipping option that enables the NV to benefit from a replenishment quantity Q_2 from a (closer) drop shipper, at a unit product purchase cost w_2. Such drop-shipped products are assumed to be provided directly to consumers' home (without transiting by store). Hence, while x_1 has to be entirely served by Q_1, x_2 can be both served by Q_1 and Q_2, as shown in Figure 4.1.

From the end consumer perspective, products can therefore be bought from the store at a unit product selling price v_1 or from Internet at a unit product selling price v_2. When products are bought on Internet, the replenishment source is either the store (when Q_1 is high enough) or the drop shipper.

Both store demand and Internet demand are subject to product returns. Indeed, a portion of products bought is assumed to be systematically returned. The return rates are assumed deterministic. Furthermore, return rates are considered to be different for different types of flows: β_1 is the return rated associated with products sold in store (products that are replenished from the distant supplier); β_2 is the return rate associated with products sold on Internet and replenished from the drop shipper; β_3 is the return rate associated with products sold on Internet and replenished from store. Practically β_1 is smaller than others because e-commerce tends to have a higher return rate than traditional commerce. $\beta_2 \geq \beta_3$ since when Internet demand is satisfied by store inventory (rather than the drop-shipper), we expect that the NV would offer a higher quality than the drop shipper in packaging, labeling delivery, and other consumer services to ensure a good consumer satisfaction which is a key element for the NV, which would reduce the return rate.

Returned products are considered to be resalable in the selling period (as new products) after a certain treatment process performed in store at a unit cost w_r that includes the delivery cost between consumer and store, product examination and control cost, an eventual repair cost, product repackaging and relabeling cost, etc. We assume that the time between the initial sale and a resale in case the product is returned is small relative to the selling season. Hence, returned products are considered as part of store inventory immediately after treatment.

Store demand x_1 is served by store inventory ordered before the season Q_1 and by product returns occurring during the season. Internet demand x_2 is served by store inventory, drop shipping option Q_2 and returns occurring during the season. In other words, the quantity Q_2 can not be used for serving store demand directly.

The unit selling price v_2 is lower than v_1 and the unit purchasing cost w_2 for drop shipping is higher than w_1, because the retailer usually needs to pay the drop-shipper a higher product unit purchase cost than to the distant supplier and in addition, the unit selling price paid by the internet consumer is expected to be lower than the price applied in the physical store. Therefore, when there is not enough inventory to satisfy both x_1 and x_2, the NV allocates store inventory to satisfy x_1 (with priority 1) and then use the remaining inventory for x_2 (with priority 2).

If a unit of product remains at store at the end of the selling season, it is assumed to be salvaged at unit price s.

In case of shortage in the store, it is assumed that a portion t of consumers switch to the drop shipping option, i.e. they become Internet consumers. For the rest of store consumers, a lost sale penalty p per unit of product is applied.

Hereafter are the additional modeling assumptions:

- Store demand and internet demand are two independent random variables. The probability distribution function of each demand is assumed to be known when ordering Q_1.

- The supply capacity of drop-shipping option is unlimited, i.e. there is no restriction on values that Q_2 can take.

- We also make the following assumption that is standard for NV Problem: $v_1 > w_1 > s$, $v_2 > w_2 > s$.

Hence, by formulating the expected profit function for the NV, Q_2 is deduced from the realizations of x_1 and x_2, while the optimal store order quantity Q_1 is determined by optimizing the expected profit.

If we eliminate the assumption on product returns, the model is equivalent to the one of [19].

Define the following notations used in Chapter 4:

x_1 the random variable representing demand at store. It is assumed to have a continuous probability function $f_1(x_1)$ and cumulative function $F_1(x_1)$, with mean μ_1 and standard deviation σ_1,

x_2 the random variable representing demand on Internet. It is assumed to have a continuous probability function $f_2(x_2)$ and cumulative function $F_2(x_2)$, with mean μ_2 and standard deviation σ_2,

β_1 return rate associated with products sold in store,

β_2 return rate associated with products replenished from drop shipper and sold on Internet,

β_3 return rate associated with products replenished from store and sold on Internet,

t proportion of consumers who accept switching from store to drop-shipping option in case of shortage in the store,

w_r unit return handling cost in the store,

v_1 unit selling price for a product bought in store,

v_2 unit selling price for a product bought on Internet,

w_1 unit purchasing price cost from the distant supplier,

w_2 unit purchasing price cost for the drop-shipping option,

p unit penalty cost of shortage when store demand is unsatisfied,

s unit discount price for store inventory when overstock happens,

Q_1 order quantity before the season, the decision variable of the model,

Q_2 drop-shipping order quantity.

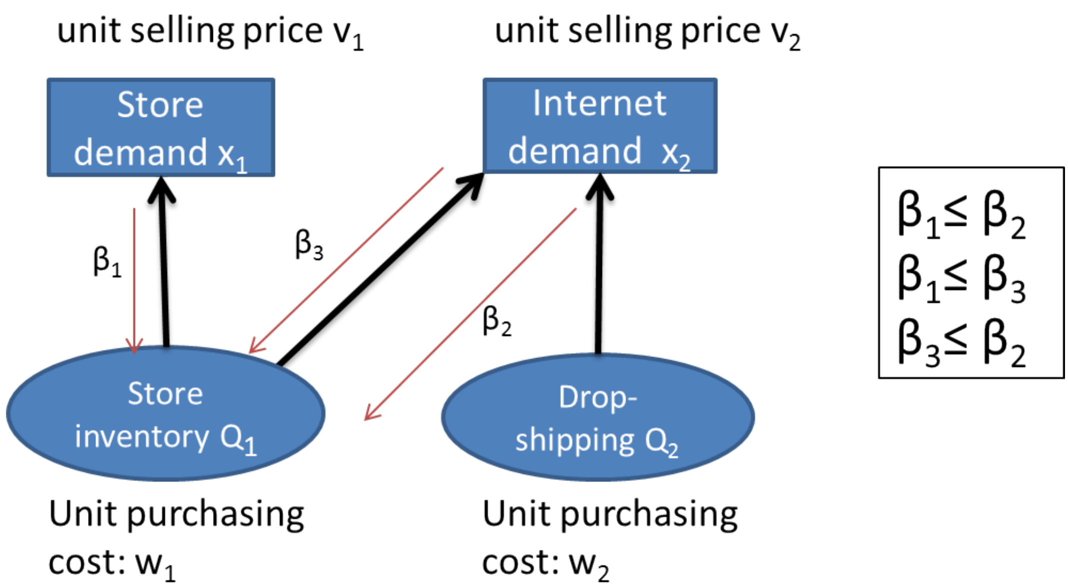


Figure 4.1: Problem modeling

4.4 Problem formulation

The mathematical formulation of the model is obtained by considering different situations that may arise regarding the inventory that is available in store (i.e. the sum of Q_1 and product returns that are used to satisfy demand after being treated in store) and the quantity Q_2 ordered from the drop-shipper (as well as the associated product returns) on one hand, and the realizations of demands x_1 and x_2 on the other hand. More specifically, we identify several cases.

The first case, i.e. Case 1 here below, corresponds to the situation where the sum of Q_1 and product returns associated with store and Internet demands is sufficient to satisfy both the realizations of x_1 and x_2.

In the second case, i.e. Case 2, the quantity Q_1 and product returns associated with store are sufficient to satisfy x_1. x_2 is then satisfied with the remaining store inventory and the quantity Q_2 ordered from the drop shipper as well as the related product returns.

The third case, i.e. Case 3, corresponds to the situation where the sum of Q_1 and product returns associated with store demand are not sufficient to satisfy x_1. Depending on the assumption considered, we identify two variants of models. In Model 1, we assume that product returns associated with Internet sales cannot be used to satisfy x_1. Thus, only product returns associated with store can be used to satisfy x_1 (this assumption can be seen in [77]). In Model 2, we relax this assumption by considering that both types of product returns (store and Internet) can be used to satisfy x_1.

Note that in the variants of models, store demand x_1 is assumed to be satisfied in priority compared to Internet demand x_2.

To sum up, two variants of model, i.e. Model 1 and Model 2, can be formulated depending on whether returns associated with Internet sales can be used for satisfying x_1. In the following, we give the formulations of both variants. Firstly, we formulate the elementary profits associated with Case 1 and 2 that are common to Model 1 and 2. Then section 4.4.1 gives the formulation of the complete expected profit pertaining to Model 1. Section 4.4.2 gives the formulation of the complete expected profit pertaining to Model 2.

Case 1: the store inventory at the beginning of the selling season i.e. Q_1, together with product returns is enough to satisfy both x_1 and x_2. In this situation, the NV needs no drop-shipping. We denote the realized demand at store as X_1, then the associated return is $X_1\beta_1$, thus the net sale related to X_1 is $X_1(1-\beta_1)$. With the same logic, the net sale related to the realized internet demand X_2 is $X_2(1-\beta_3)$ and the related return is $X_2\beta_3$. Obviously, Q_1 should not be smaller than the total net sale, thus the condition for case 1 is:

$$Q_1 \geqslant x_1(1-\beta_1) + x_2(1-\beta_3)$$

Case 2: the realized demands X_1 and X_2 can not be entirely satisfied by Q_1. Store inventory is first used to satisfy store demand X_1, thus the net store sale is $X_1(1-\beta_1)$ and the related return is $X_1\beta_1$. The NV uses the rest of store inventory i.e. $Q_1 - X_1(1-\beta_1)$ as well the drop-shipped quantity Q_2 to satisfy X_2. We have

$$X_2 = \frac{Q_1 - X_1(1-\beta_1)}{1-\beta_3} + (Q_2 + \frac{Q_2\beta_2}{1-\beta_3})$$

which gives

$$Q_2 = \frac{X_2(1-\beta_3) + X_1(1-\beta_1) - Q_1}{1-\beta_3+\beta_2}$$

The net sale on Internet is thus $(Q_1 - X_1(1-\beta_1)) + Q_2$ and the related return is $\frac{Q_2\beta_2}{1-\beta_3} + \frac{(Q_1 - x_1(1-\beta_1))\beta_3}{1-\beta_3}$.

Q_1 should be larger than the net sale related to X_1, and Q_2 should be positive. Thus the condition for case 2 is:

$$x_1(1-\beta_1) + x_2(1-\beta_3) > Q_1 \geqslant x_1(1-\beta_1)$$

4.4.1 Model 1

In this model, the return associated with x_2 can not be used to satisfy x_1. In case 3, the sum of Q_1 and product returns associated with store sales are not sufficient to satisfy x_1. Figure 4.2 displays the areas associated with Case 1, 2 and 3 as a function of x_1 and x_2.

Case 3, demand x_1 is larger than the store inventory Q_1 and the associated product returns: $Q_1 < x_1(1-\beta_1)$. Thus the store sale equals to Q_1 and the related return is $\frac{Q_1\beta_1}{1-\beta_1}$. A portion of store consumers switch to drop-shipping option when there is no more inventory in store, i.e. the unsatisfied store demand $X_1 - \frac{Q_1}{1-\beta_1}$ is partly

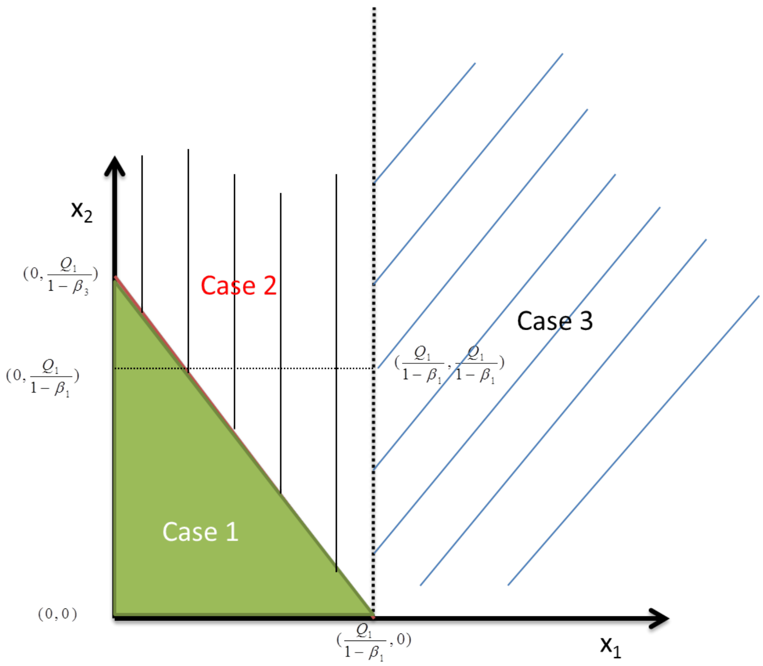


Figure 4.2: 3 cases for model 1 as the realized values X_1 and X_2 change

transferred to Internet demand. This new Internet demand transferred from store demand is denoted as $X_2^{'}$: $X_2^{'} = t(X_1 - \frac{Q_1}{1-\beta_1})$. The rest is lost with a penalty cost: $(1-t)p(X_1 - \frac{Q_1}{1-\beta_1})$.

Q_2 is ordered to the drop-shipper and $Q_2\beta_2$ products are returned:

$$Q_2\beta_2 = (X_2 + X_2^{'} - Q_2)(1-\beta_3) \tag{4.1}$$

Thus

$$Q_2 = \frac{(X_2 + X_2')(1-\beta_3)}{1-\beta_3+\beta_2}$$

The net Internet sale equals to Q_2 and the related return is $\frac{Q_2\beta_2}{1-\beta_3}$.

case	sales realized in store	return related to X_1	sale realized on Internet	return related to X_2
1	$X_1(1-\beta_1)$	$X_1\beta_1$	$X_2(1-\beta_3)$	$X_2\beta_3$
2	$X_1(1-\beta_1)$	$X_1\beta_1$	$Q_2 + Q_1 - X_1(1-\beta_1)$	$\frac{Q_2\beta_2}{1-\beta_3} + \frac{(Q_1-C_1(1-\beta_1))\beta_3}{1-\beta_3}$
3	Q_1	$\frac{Q_1\beta_1}{1-\beta_1}$	Q_2	$\frac{Q_2\beta_2}{1-\beta_3}$

Table 4.1: Total sale and return for 3 cases in model 1

Total sale and return for different cases are shown in Table 4.1. One condition needs to be validated: the revenue related to x_2 is larger than the return cost, otherwise it is not profitable to reuse the returned products.

$$(Q_2 + Q_1 - x_1(1-\beta_1))(v_2 - w_2) > \frac{Q_2\beta_2}{1-\beta_3}w_r + \frac{(Q_1 - x_1(1-\beta_1))\beta_3}{1-\beta_3}w_r$$

As a result,

$$Q_2(v_2 - w_2) > \frac{Q_2\beta_2}{1-\beta_3}w_r \Rightarrow v_2 - w_2 > \frac{\beta_2}{1-\beta_3}w_r$$

The profit function is derived as in equation 4.2.

$$\pi = \begin{cases} v_1x_1(1-\beta_1) + v_2x_2(1-\beta_3) - w_1Q_1 + s(Q_1 - x_1(1-\beta_1) - x_2(1-\beta_3)) - (x_1\beta_1 + x_2\beta_3)w_r & \text{case 1} \\ v_1x_1(1-\beta_1) + v_2(Q_1 - x_1(1-\beta_1) + Q_2) - w_1Q_1 - w_2Q_2 - (x_1\beta_1 + \frac{Q_2\beta_2}{1-\beta_3} + \frac{(Q_1-x_1(1-\beta_1))\beta_3}{1-\beta_3})w_r & \text{case 2} \\ v_1Q_1 + v_2Q_2 - w_1Q_1 - w_2Q_2 - (1-t)p(x_1 - \frac{Q_1}{1-\beta_1}) - (\frac{Q_1}{1-\beta_1}\beta_1 + \frac{Q_2}{1-\beta_3}\beta_2)w_r & \text{case 3} \end{cases} \tag{4.2}$$

Proposition 1. *The expected profit is concave when* $\beta_2 \geq \beta_3 - \frac{w_2-s}{v_2+w_r-s}$.

Proof. Proof is provided in Appendix 1. □

Practically $\beta_2 \geq \beta_3$, meanwhile, it is a key element for the NV to ensure a good consumer satisfaction and as he is in direct communication with consumers, he makes more efforts than the drop shipper in packaging, labeling delivery, and other consumer services. Thus $\beta_2 \geq \beta_3 - \frac{w_2-s}{v_2+w_r-s}$ and thus the concavity is validated.

The optimal condition is derived by setting the equation bellow (equation 6.27 of Appendix 1) equal to 0:

$$\lambda_2F_1(\frac{Q_1}{1-\beta_1}) + \lambda_3(1 - F_1(\frac{Q_1}{1-\beta_1})) + \int_{-\infty}^{\frac{Q_1}{1-\beta_1}} (\lambda_1 - \lambda_2)F_2(\frac{Q_1}{1-\beta_3} - \frac{1-\beta_1}{1-\beta_3}x_1)f_1(x_1)dx_1 = 0 \tag{4.3}$$

See λ_1, λ_2 and λ_3 in Appendix 1.

Proposition 2. The optimal expected profit function is derived as:

$$\begin{aligned} E(\pi(Q_1^*)) = \alpha_3 \int_{\frac{Q_1^*}{1-\beta_1}}^{\infty} x_1 f(x_1)dx_1 + \alpha_2 \int_{-\infty}^{\frac{Q_1^*}{1-\beta_1}} x_1 f(x_1)dx_1 + b_2\mu_2 \\ + (b_1 - b_2)\int_{-\infty}^{\frac{Q_1^*}{1-\beta_1}} \int_{-\infty}^{\frac{Q_1^*}{1-\beta_3} - \frac{1-\beta_1}{1-\beta_3}x_1} (\frac{1-\beta_1}{1-\beta_3}x_1 + x_2)f_2(x_2)dx_2f_1(x_1)dx_1 \end{aligned} \tag{4.4}$$

See α_1, α_2, b_1 and b_2 in Appendix 1.

Proof. Proof is provided in Appendix 2. □

The above results are the formulations for general situations, but in practice, some special cases can happen:

4.4.1.1 Special case: $\beta_1 = \beta_2 = \beta_3 = 0$

When we do not consider the return effect, $\beta_1 = \beta_2 = \beta_3 = 0$, equation 6.27 is derived as:

$$v_1 - w_1 + p - (v_2 - w_2 + p)t - (v_1 - w_2 + p - (v_2 - w_2 + p)t)F_1(Q_1^*) - (w_2 - s)F_{1+2}(Q_1^*) \quad (4.5)$$

Equation 4.5 can be adapted to general demand distributions. We will find same results as in [19], which developed a NV problem with drop-shipping option but no product returns.

4.4.1.2 Special case: $\beta_1 = \beta_2 = \beta_3 = \beta$

This special case assumes that all return rates are identical. In this situation, our model is greatly simplified. The expected profit is concave and the optimal order quantity is developed as:

$$(w_2 - w_1)F_1(\frac{Q_1}{1-\beta_1}) + (v_1 - w_1 - v_2 t + w_2 t + \frac{\beta w_r}{1-\beta}(t-1) + \frac{1-t}{1-\beta}p)(1 - F_1(\frac{Q_1}{1-\beta_1})) + (w_2 - s)F_{1+2}(\frac{Q_1}{1-\beta}) = 0 \quad (4.6)$$

The optimal expected profit is:

$$\begin{aligned} E(\pi(Q_1^*)) = &((1-\beta)(v_2 - w_2)t - \beta w_r t - (1-t)p) \int_{\frac{Q_1^*}{1-\beta}}^{\infty} x_1 f(x_1) dx_1 \\ &+ ((v_1 - w_2)(1-\beta) - w_r \beta) \int_{-\infty}^{\frac{Q_1^*}{1-\beta}} x_1 f(x_1) dx_1 + ((v_2 - w_2)(1-\beta) - w_r \beta)\mu_2 \\ &+ (w_2 - s)(1-\beta) \int_{-\infty}^{\frac{Q_1^*}{1-\beta}} \int_{-\infty}^{\frac{Q_1^*}{1-\beta} - \frac{1-\beta}{1-\beta}x_1} (x_1 + x_2) f_2(x_2) dx_2 f_1(x_1) dx_1 \end{aligned} \quad (4.7)$$

4.4.1.3 Special case: $x_1 = 0$

This is the case where the NV is a pure e-retailer without a store. Companies like Amazon put the entire consumer experience - from browsing products to placing orders to paying for purchases - on the Internet. The NV has also two options of supplying: to pass an order to the supplier before the selling season and drop-shipping option during the season. Considering the return effect, the problem can be treated as a special case of our model with zero store demand. The optimal order quantity condition is developed as:

$$\lambda_2 + (\lambda_1 - \lambda_2)F_2(\frac{Q_1^*}{1-\beta_3}) = 0 \tag{4.8}$$

thus

$$F_2(\frac{Q_1^*}{1-\beta_3}) = \frac{(\beta_2-\beta_3)(v_2+w_r)+w_2-w_1(1+\beta_2-\beta_3)}{(\beta_2-\beta_3)(v_2+w_r)+w_2-s(1+\beta_2-\beta_3)} \tag{4.9}$$

When $\beta_2 \geq \beta_3 - \frac{w_2-s}{v_2+w_r-s}$, the expected profit is concave; if not, the optimal order quantity $Q_1^* = 0$.

And the optimal expected profit is developed as:

$$\begin{aligned} E(\pi(Q_1^*)) &= b_2\mu_2 + (b_1-b_2)\int_{-\infty}^{\frac{Q_1^*}{1-\beta_3}} x_2 f_2(x_2)dx_2 \\ &= (v_2 - w_2 - \frac{\beta_2 w_r}{1-\beta_3})\frac{1-\beta_3}{1-\beta_3+\beta_2}\mu_2 \\ &+ \frac{1-\beta_3}{1-\beta_3+\beta_2}(w_2 - s + \frac{\beta_2 w_r}{1-\beta_3} - \frac{\beta_3 w_r}{1-\beta_3})\int_{-\infty}^{\frac{Q_1^*}{1-\beta_3}} x_2 f_2(x_2)dx_2 \end{aligned} \tag{4.10}$$

When $\beta_2 = \beta_3$, the optimal order quantity and expected profit is derived as:

$$F_2(\frac{Q_1^*}{1-\beta_3}) = \frac{w_2-w_1}{w_2-s} \tag{4.11}$$

$$E(\pi(Q_1^*)) = (v_2 - w_2 - \frac{\beta_3 w_r}{1-\beta_3})(1-\beta_3)\mu_2 + (1-\beta_3)(w_2-s)\int_{-\infty}^{\frac{Q_1^*}{1-\beta_3}} x_2 f_2(x_2)dx_2 \tag{4.12}$$

Let us note that equation 4.9 is analogue to the optimality condition for a NV with an emergency option derived by [18], if we consider drop-shipping as a special emergency option. Since $1-\beta_3 < 1$, $Q_1^* = (1-\beta_3)F_2^{-1}(\frac{w_2-w_1}{w_2-s}) < F_2^{-1}(\frac{w_2-w_1}{w_2-s})$. In other words, the optimal order quantity in presence of product returns is smaller than the one of the NV problem with an emergency option but no product returns. Such a result is intuitive because in the presence of product returns, some demand does not result in a real sale because the product is returned and the price is payed back to the consumer, thus less inventory is required. If $\beta_2 = \beta_3 = 0$, equation 4.9 is identical to the one in [18]. When $w_2 = w_2$, equation 4.9 gives $Q_1^* = 0$. This is the special situation that the two supply options have same purchasing cost and same return rate, there will be no longer any interest to stock an initial store inventory.

4.4.1.4 Special case: no drop-shipping option

This special case assume that no drop-shipping option is available for the NV. This is a NVP with two independent demands and product returns. In this situation, as the demands realize, we have same 3 cases as in model 1 but with $Q_2 = 0$, the profit function is derived as in equation 4.13:

$$\pi = \begin{cases} v_1x_1(1-\beta_1) + v_2x_2(1-\beta_3) - w_1Q_1 + s(Q_1 - x_1(1-\beta_1) - x_2(1-\beta_3)) - (x_1\beta_1 + x_2\beta_3)w_r & \text{case 1} \\ v_1x_1(1-\beta_1) + v_2(Q_1 - x_1(1-\beta_1)) - w_1Q_1 - (x_1\beta_1 + \frac{(Q_1 - x_1(1-\beta_1))\beta_3}{1-\beta_3})w_r & \text{case 2} \\ v_1Q_1 - w_1Q_1 - p(x_1 - \frac{Q_1}{1-\beta_1}) - \frac{Q_1}{1-\beta_1}\beta_1 w_r & \text{case 3} \end{cases} \tag{4.13}$$

The first derivative can be derived as:

$$\begin{aligned} \frac{dE(\pi(Q_1))}{dQ_1} &= v_1 - w_1 + \frac{p}{1-\beta_1} - \frac{\beta_1}{1-\beta_1}w_r \\ &+ (v_2 - \frac{\beta_3 w_r}{1-\beta_3} + \frac{\beta_1 w_r}{1-\beta_1} - v_1 - \frac{p}{1-\beta_1})F_1(\frac{Q_1}{1-\beta_1}) \\ &+ (s - v_2 + \frac{\beta_3 w_r}{1-\beta_3})\int_{-\infty}^{\frac{Q_1}{1-\beta_1}} F_2(\frac{Q_1}{1-\beta_3} - \frac{1-\beta_1}{1-\beta_3}x_1)f_1(x_1)dx_1 \end{aligned} \tag{4.14}$$

Proposition 3. The expected profit is concave.

It is easy to prove that the second derivative is negative, thus the expected profit is concave. The optimal condition is derived by setting equation 4.14 equals 0.

4.4.2 Model 2

In this model, case 3 has two subcases: case 3a and 3b, see Figure 4.3. In case 3a, the sum of quantity Q_1 and product returns associated with store are not sufficient to satisfy x_1, but the sum of quantity Q_1 and product returns associated with store and from Internet are sufficient to satisfy x_1. In case 3b, the sum of quantity Q_1 and product returns associated with store and from Internet are not sufficient to satisfy x_1.

Case 3a: the demand X_1 is larger than the store inventory Q_1 and the associated product returns: $Q_1 < X_1(1-\beta_1)$. Thus the NV uses returned products from internet sales for unsatisfied part of X_1. Q_2 is ordered to the drop-shipper and $Q_2\beta_2$ products are returned, which is partially used to serve X_1: $X_1(1-\beta_1) - Q_1$. Then the net store

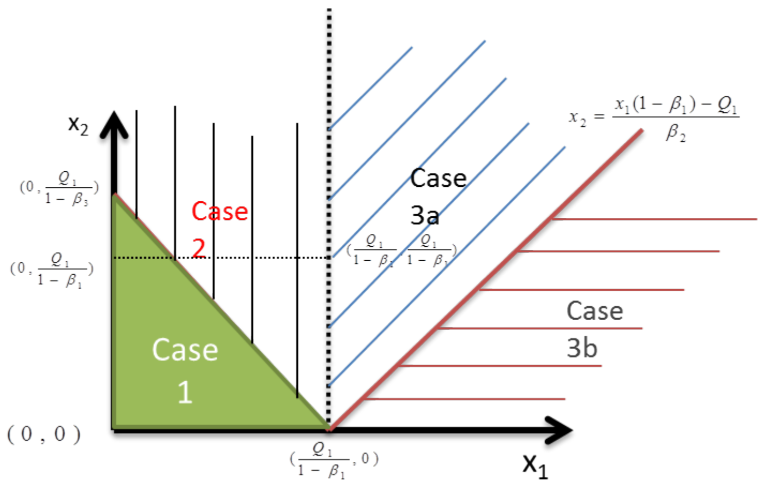


Figure 4.3: 3 cases for model 2 as the realized values X_1 and X_2 change

sale is $X_1(1-\beta_1)$ and the related return is $x_1\beta_1$. The other part $Q_2\beta_2-(X_1(1-\beta_1)-Q_1)$ is used for $X_2 - Q_2$:

$$\frac{Q_2\beta_2 - (X_1(1-\beta_1)-Q_1)}{1-\beta_3} = X_2 - Q_2 \tag{4.15}$$

Thus the net sale on Internet is $Q_2 - X_1(1-\beta_1) + Q_1$ and the related return is $\frac{Q_2\beta_2}{1-\beta_3} + \frac{(Q_1-X_1(1-\beta_1))\beta_3}{1-\beta_3}$.

$$Q_2 = \frac{X_2(1-\beta_3) + X_1(1-\beta_1) - Q_1}{1-\beta_3+\beta_2}$$

$Q_1 < X_1$ and $Q_2\beta_2 - (X_1(1-\beta_1) - Q_1) > 0$. Thus the condition for case 3a is:

$$x_1(1-\beta_1) > Q_1 \geqslant x_1(1-\beta_1) - x_2\beta_2$$

Case 3b: the demand X_1 is larger than the store inventory Q_1 and the associated product returns, the NV uses all returned products from internet sale for the rest part of X_1. In this situation, the NV passes an maximal internet order to the drop shipper: $Q_2 = X_2$, then the returned product is $Q_2\beta_2$ which is all used for store demand X_1. Then the net sale in store is $Q_1 + Q_2\beta_2$ and the related return is $\frac{Q_1+Q_2\beta_2}{1-\beta_1}\beta_1$. The net sale related to X_2 is $Q_2 - Q_2\beta_2$ and the related return is $Q_2\beta_2$.

As we know, X_1 is not totally satisfied, we have $\frac{Q_1+Q_2\beta_2}{1-\beta_1} < X_1$. Then the condition for case 3b is:

$$x_1(1-\beta_1) - x_2\beta_2 > Q_1$$

Considering that store consumers may switch to drop-shipping option when there is no inventory in store, the unsatisfied store demand $X_1 - \frac{Q_1+Q_2\beta_2}{1-\beta_1}$ is partly transferred to internet demand. This Internet demand transferred from store demand is denoted as $X_2^{'}$: $X_2^{'} = t(X_1 - \frac{Q_1+Q_2\beta_2}{1-\beta_1})$. The rest demand is lost with a unit penalty cost: $(1-t)p(X_1 - \frac{Q_1+Q_2\beta_2}{1-\beta_1})$.

An order of $Q_2^{'}$ is passed by the NV to drop shipper for satisfying $x_2^{'}$: $X_2^{'} = Q_2^{'} + \frac{Q_2^{'}\beta_2}{1-\beta_3}$. We have the net sale related to $X_2^{'}$ is $Q_2^{'}$ and the related return equals to $\frac{\beta_2 Q_2^{'}}{1-\beta_3}$.

$$Q_2^{'} = t(X_1 - \frac{Q_1+Q_2\beta_2}{1-\beta_1})\frac{1-\beta_3}{1-\beta_3+\beta_2}$$

case	sale in store	return related to X_1	sale related to X_2	return related to X_2	sale related to $X_2^{'}$	return related to $X_2^{'}$
1	$X_1(1-\beta_1)$	$X_1\beta_1$	$X_2(1-\beta_3)$	$X_2\beta_3$	0	0
2	$X_1(1-\beta_1)$	$X_1\beta_1$	$Q_2 + Q_1 - X_1(1-\beta_1)$	$\frac{Q_2\beta_2}{1-\beta_3} + \frac{(Q_1-X_1(1-\beta_1))\beta_3}{1-\beta_3}$	0	0
3a	$X_1(1-\beta_1)$	$X_1\beta_1$	$Q_2 + Q_1 - X_1(1-\beta_1)$	$\frac{Q_2\beta_2}{1-\beta_3} + \frac{(Q_1-X_1(1-\beta_1))\beta_3}{1-\beta_3}$	0	0
3b	$Q_1 + Q_2\beta_2$	$\frac{Q_1+Q_2\beta_2}{1-\beta_1}\beta_1$	$Q_2 - Q_2\beta_2$	$Q_2\beta_2$	$Q_2^{'}$	$\frac{\beta_2 Q_2^{'}}{1-\beta_3}$

Table 4.2: Total sale and return for 4 cases in model 2

Total sale and return for different cases are shown in Table 4.2. One condition needs to be validated: the revenue related to x_2 is larger than the return cost, otherwise it is not profitable to reuse the returned products.

$$(Q_2 + Q_1 - x_1(1-\beta_1))(v_2 - w_2) > \frac{Q_2\beta_2}{1-\beta_3}w_r + \frac{(Q_1 - x_1(1-\beta_1))\beta_3}{1-\beta_3}w_r$$

This should be satisfied for both Case 2 and 3. As a result,

$$Q_2(v_2 - w_2) > \frac{Q_2\beta_2}{1-\beta_3}w_r \Rightarrow v_2 - w_2 > \frac{\beta_2}{1-\beta_3}w_r$$

The profit function is derived as in equation 4.16.

Proposition 4. The expected profit is concave when $\beta_2 \geq \beta_3 - \frac{w_2-s}{v_2+w_r-s}$.

Proof. Proof is provided in Appendix 3. $\square$

$$\pi = \begin{cases} v_1x_1(1-\beta_1) + v_2x_2(1-\beta_3) - w_1Q_1 + s(Q_1 - x_1(1-\beta_1) - x_2(1-\beta_3)) - (x_1\beta_1 + x_2\beta_3)w_r & \text{case 1} \\ v_1x_1(1-\beta_1) + v_2(Q_1 - x_1(1-\beta_1) + Q_2) - w_1Q_1 - w_2Q_2 - (x_1\beta_1 + \frac{Q_2\beta_2}{1-\beta_3} + \frac{(Q_1-x_1(1-\beta_1))\beta_3}{1-\beta_3})w_r & \text{case 2} \\ v_1x_1(1-\beta_1) + v_2(Q_1 - x_1(1-\beta_1) + Q_2) - w_1Q_1 - w_2Q_2 - (x_1\beta_1 + \frac{Q_2\beta_2}{1-\beta_3} + \frac{(Q_1-x_1(1-\beta_1))\beta_3}{1-\beta_3})w_r & \text{case 3a} \\ v_1(Q_1 + Q_2\beta_2) + v_2Q_2(1-\beta_2) - w_1Q_1 - w_2Q_2 + (v_2 - w_2)Q_2^{'} & \\ -(x_1 - \frac{Q_1+Q_2\beta_2}{1-\beta_1})(1-t)p - (\frac{Q_1}{1-\beta_1}\beta_1 + \frac{Q_2}{1-\beta_1}\beta_2 + \frac{Q_2^{'}}{1-\beta_3}\beta_2)w_r & \text{case 3b} \end{cases} \tag{4.16}$$

Practically $\beta_2 \geq \beta_3$, meanwhile, it is a key element for the NV to ensure a good consumer satisfaction and as he is in direct communication with consumers, he makes more efforts than the drop shipper in packaging, labeling delivery, and other consumer services. Thus $\beta_2 \geq \beta_3 - \frac{w_2-s}{v_2+w_r-s}$ and thus the concavity is validated.

The optimal condition is derived as :

$$\begin{aligned} &\lambda_3 + \int_{\frac{Q_1^*}{1-\beta_1}}^{\infty} (\lambda_4 - \lambda_3)F_2(\frac{x_1(1-\beta_1) - Q_1^*}{\beta_2})f_1(x_1)dx_1 \\ &\quad + \int_{-\infty}^{\frac{Q_1^*}{1-\beta_1}} (\lambda_1 - \lambda_2)F_2(\frac{Q_1^*}{1-\beta_3} - \frac{1-\beta_1}{1-\beta_3}x_1)f_1(x_1)dx_1 = 0 \end{aligned} \tag{4.17}$$

See λ_1, λ_2, λ_3 and λ_4 in Appendix 3.

Proposition 5. The optimal expected profit function (c.f. Appendix 4) is:

$$\begin{aligned} E(\pi(Q_1^*)) = \alpha_2\mu_1 + b_2\mu_2 + (b_1 - b_2) \int_{-\infty}^{\frac{Q_1^*}{1-\beta_1}} \int_{-\infty}^{\frac{Q_1^*}{1-\beta_3} - \frac{1-\beta_1}{1-\beta_3}x_1} &(\frac{1-\beta_1}{1-\beta_3}x_1 \\ &+ x_2)f_2(x_2)dx_2f_1(x_1)dx_1 \\ + (b_4 - b_3) \int_{\frac{Q_1^*}{1-\beta_1}}^{\infty} \int_{-\infty}^{\frac{x_1(1-\beta_1)-Q_1^*}{\beta_2}} (-\frac{1-\beta_1}{\beta_2}x_1 + x_2)f_2(x_2)dx_2f_1(x_1)dx_1 \end{aligned} \tag{4.18}$$

See α_2, b_1, b_2, b_3 and b_4 in Appendix 3.

Some special cases happen in practice:

4.4.2.1 Special case: $\beta_1 = \beta_2 = \beta_3 = 0$

When we do not consider the return effect, $\beta_1 = \beta_2 = \beta_3 = 0$, equation 4.17 is derived as:

$$v_1 - w_1 + p - (v_2 - w_2 + p)t - (v_1 - w_2 + p - (v_2 - w_2 + p)t)F_1(Q_1^*) - (w_2 - s)F_{1+2}(Q_1^*) \tag{4.19}$$

As in Model 1, Equation 4.19 can be adapted to general demand distributions. We find same results as in [19].

4.4.2.2 Special case: $\beta_1 = \beta_2 = \beta_3 = \beta$

This special case assume that all return rates are identical. In this situation, the expected profit is concave and the optimal order quantity is developed as:

$$F_{1+2}\left(\frac{Q_1^*}{1-\beta}\right) - \frac{w_2 - w_1}{w_2 - s} - \frac{v_1 - w_2 - \frac{w_r\beta}{1-\beta} - \frac{1}{1-\beta}((1-\beta)(v_2 - w_2)t - \beta w_r t - (1-t)p)}{w_2 - s} \int_{\frac{Q_1^*}{1-\beta}}^{\infty} F_2\left(\frac{x_1(1-\beta) - Q_1^*}{\beta}\right) f_1(x_1)dx_1 = 0 \tag{4.20}$$

The optimal expected profit is:

$$\begin{aligned} E(\pi(Q_1^*)) = \alpha_2\mu_1 + b_2\mu_2 + (b_1 - b_2) \int_{-\infty}^{\frac{Q_1^*}{1-\beta}} \int_{-\infty}^{\frac{Q_1^*}{1-\beta} - x_1} (x_1 + x_2) f_2(x_2)dx_2 f_1(x_1)dx_1 \\ + (b_4 - b_3) \int_{\frac{Q_1^*}{1-\beta}}^{\infty} \int_{-\infty}^{\frac{x_1(1-\beta) - Q_1^*}{\beta}} \left(-\frac{1-\beta}{\beta} x_1 + x_2\right) f_2(x_2)dx_2 f_1(x_1)dx_1 \end{aligned} \tag{4.21}$$

4.4.2.3 Special case: $\mu_1 = 0$, $\sigma_1 = 0$

This is the case where the NV is a pure e-retailer without a physical store. We have same results as in model 1 since when $x_1 = 0$, the assumption that return products from Internet sale can be used for store demand does not make sense.

4.4.2.4 Special case: no drop-shipping option

We have same results as in model 1 since when $Q_2 = 0$, the assumption that return products from Internet sale satisfied by drop-shipping can be used for store demand does not make sense.

The contrary to the above situations is that the NV does not offer a higher quality than the drop-shipper satisfying Internet demand. Thus the proportion of return is larger than drop-shipping option ($\beta_3 > \beta_2$). We consider it as a extreme case and will show some insights on it by numerical examples in Appendix 5. The concavity of the expected profit is no longer guaranteed and equation 4.3 becomes the necessary optimal condition.

4.5 Numerical examples

Since Model 1 and 2 give similar results, we concentrate on Model 2 in this section. We use normally distributed demand in our examples. Other demand distributions will also work. Consider an example of a NV selling an item with the following parameter values shown in Table 4.3 (the first 6 parameters are the same as [19]). The aim of this section is threefold: 1) to evaluate the impact of the parameters on the optimal order quantity Q_1^* and the optimal expected profit $E(\pi(Q_1^*))$; 2) to identify the impact of ignoring the product returns when the NV makes decisions; 3) to identify the benefit of the drop-shipping option. Q_1^* and $E(\pi(Q_1^*))$ are derived by equations 4.17 and 4.18.

parameter	value/unit
v_1, unit selling price for an store demand	50
v_2, unit selling price for an drop-shipping demand	45
w_1, unit purchasing cost for the store order	20
w_2, unit purchasing price cost for the drop-shipping option	21
s, unit discount selling price for store inventory	10
p, unit shortage penalty cost	5
t, substitution fraction	30%
w_r, unit return cost	10
β_1, return rate associated with products sold in store	0.1
β_2, return rate associated with products replenished from drop shipper and sold on Internet	0.3
β_3, return rate associated with products replenished from store and sold on Internet	0.2

Table 4.3: Data for the numerical examples

4.5.1 Impact of $w_1, w_2, w_r, s, \beta_1, \beta_2, \beta_3$

In this part, we take an example with $\mu_1 = 100$, $\mu_2 = 10$ and $cv = 0.1, 0.2.0.3$. The impact of parameters $w_1, w_2, w_r, s, \beta_1, \beta_2, \beta_3$ are similar for other demand settings. In section 4.5.2, we will show the impact of other parameters that are not the same for different demand settings.

The impact of increasing values of w_1, w_2 and s on Q_1^* and $E(\pi(Q_1^*))$ are as expected intuitively. In particular, Figure 4.4 shows that Q_1^* and $E(\pi(Q_1^*))$ decrease with w_1. Second, on Figure 4.5, we observe that when w_2 increases, the NV tends to increase Q_1^* in order to reduce the order of drop-shipping. $E(\pi(Q_1^*))$ decreases also with w_2. Finally, when the unit salvage value s increases, as expected, Q_1^* and $E(\pi(Q_1^*))$ both increase (cf. Figure 4.6).

The impact of parameters w_r, β_1, β_2 and β_3 that are relative to product returns are represented on Figures 4.7, 4.8, 4.9 and 4.10. Q_1^* increases with w_r. The reason is that when w_r is bigger, the NV wants to reduce product returns. Since the return rate related to drop-shipping is bigger than the one related to store inventory ($\beta_3 < \beta_2$), to reduce the return cost, the NV uses more store inventory. $E(\pi(Q_1^*))$ decreases with w_r due to the fact that the return cost rises, cf. Figure 4.7.

Q_1^* decreases with β_1, because the net store demand (the difference between the "initial" consumers' demand and product returns) decreases with β_1. $E(\pi(Q_1^*))$ decreases too, because net demand decreases and return cost increases, cf. Figure 4.8.

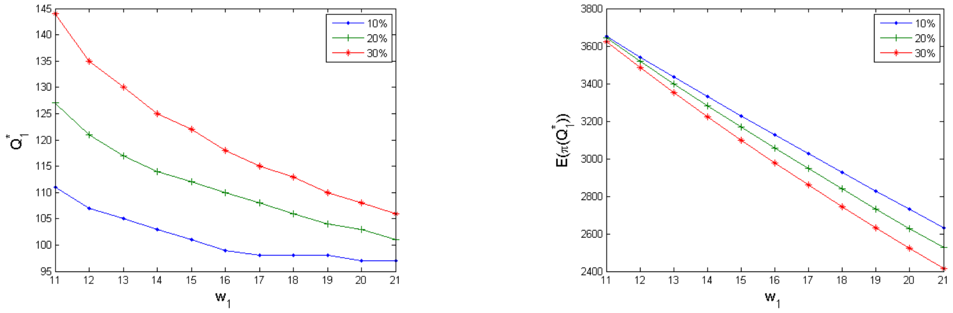
Q_1^* increases with β_2. The reason is that when β_2 increases, the NV reduces Q_2, thus the NV uses more store inventory for satisfying Internet demand. $E(\pi(Q_1^*))$ decreases with β_2 for the same reason of β_1, cf. Figure 4.9.

Q_1^* decreases with β_3. This is because when β_3 is bigger, there will be more possible return products when the NV satisfies internet demand using store inventory (including the returned products that can be reused). As a result, the NV reduces Q_1^* and orders more from drop-shipper. The expected profit decreases with β_3 for the same reason of β_1, cf. Figure 4.10.

$E(\pi(Q_1^*))$ decreases slower with β_2 than others. The reason is that when β_2 increases, the net Internet demand decreases, thus the NV tends to increase Q_1^* to satisfy more Internet demand by store inventory, as a result, the quantity of drop-shipping is reduced and the influence of β_2 is weaken. $E(\pi(Q_1^*))$ decreases faster with β_1 than the others, because when β_1 increases, the NV can not use drop-shipping to satisfy store demand in order to reduce returns related to store sale.

4.5.2 impact of v_1, v_2, p, t

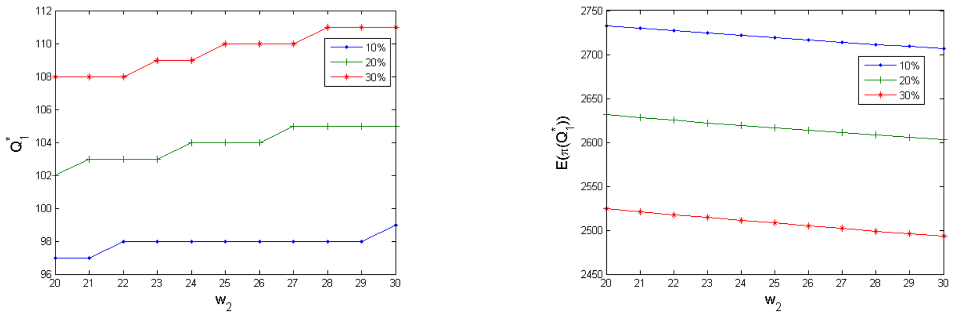
Q_1^* and $E(\pi(Q_1^*))$ both increase with μ_2, see Figure 4.11. This is because when Internet demand is bigger, on one hand, the NV increases the order quantity because there is



(a) Optimal order quantity as a function of unit store selling price

(b) Optimal expected profit as a function of unit store inventory purchasing cost

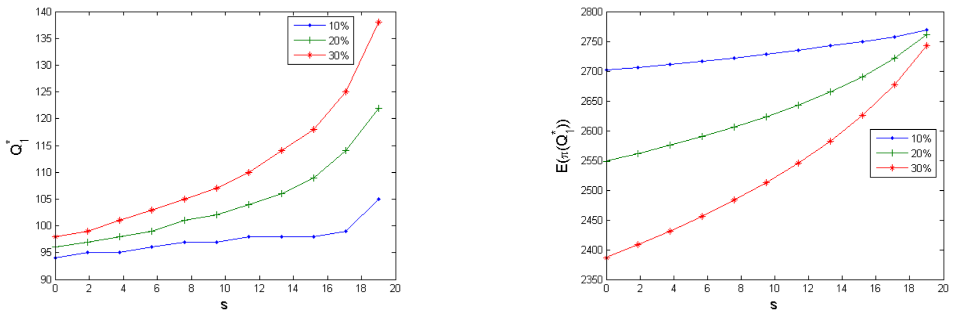
Figure 4.4: impact of w_1



(a) Optimal order quantity as a function of unit drop-shipping purchasing cost

(b) Optimal expected profit as a function of unit drop-shipping purchasing cost

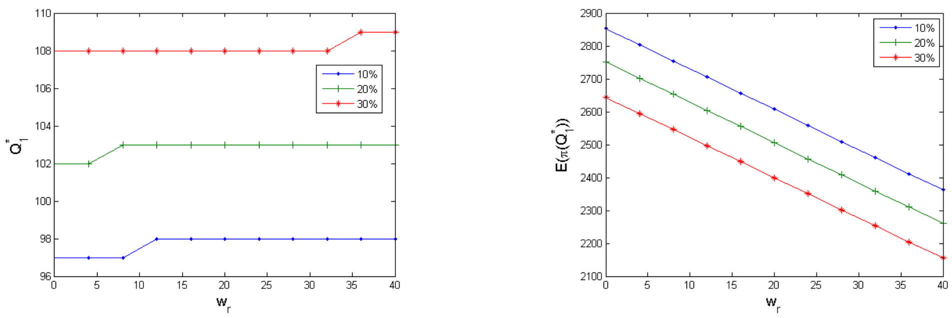
Figure 4.5: impact of w_2



(a) Optimal order quantity as a function of unit salvage value

(b) Optimal expected profit as a function of unit salvage value

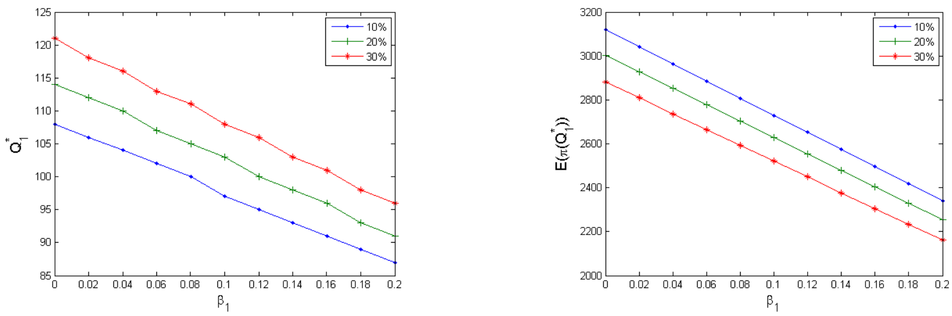
Figure 4.6: impact of s



(a) Optimal order quantity as a function of unit return cost

(b) Optimal expected profit as a function of unit return cost

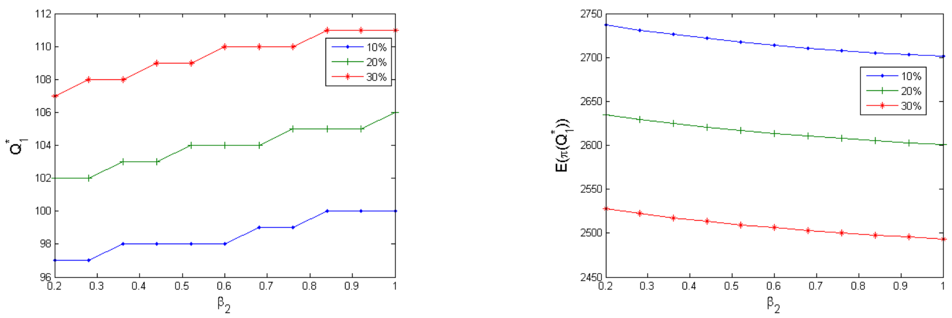
Figure 4.7: impact of w_r



(a) Optimal order quantity as a function of store sale return rate

(b) Optimal expected profit as a function of store sale return rate

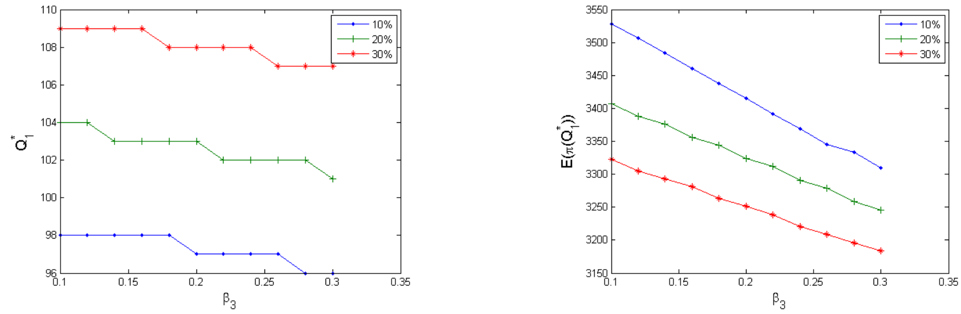
Figure 4.8: impact of β_1



(a) Optimal order quantity as a function of return rate related to Internet sale satisfied by store inventory

(b) Optimal expected profit as a function of return rate related to Internet sale satisfied by store inventory

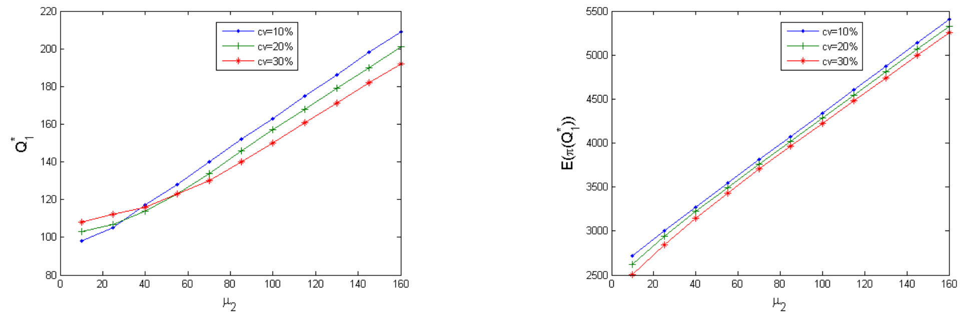
Figure 4.9: impact of β_2



(a) Optimal order quantity as a function of return rate related to Internet sale satisfied by store inventory

(b) Optimal expected profit as a function of return rate related to Internet sale satisfied by store inventory

Figure 4.10: impact of β_3

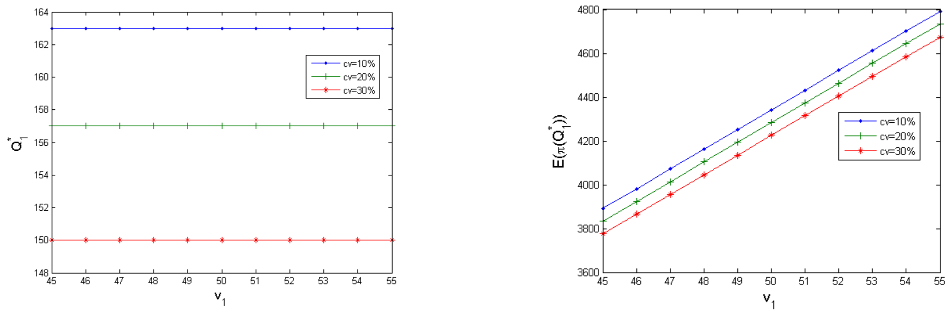


(a) Optimal order quantity as a function of unit store selling price

(b) Optimal expected profit as a function of unit store selling price

Figure 4.11: impact of μ_2

more demand, on the other hand, when Q_1 is bigger, the risk of under-stock decreases and the risk of overstock of x_1 is reduced because Q_1 can be used for satisfying x_2. Thus it is obvious that if μ_2 is big enough, Q_1^* can probably satisfy all store demand. For this reason, some results are different for different demand settings (impact of v_1, v_2, p, t): when μ_2 is bigger or close to μ_1, the results can be different from those for a small μ_2 compared with μ_1. Therefore, we give numerical analysis in two parts: one with a big μ_2 compared with μ_1 ($\mu_1 = 100$, $\mu_2 = 100$) and the other one with a small μ_2 ($\mu_1 = 100$, $\mu_2 = 10$).



(a) Optimal order quantity as a function of unit store selling price

(b) Optimal expected profit as a function of unit store selling price

Figure 4.12: impact of v_1

4.5.2.1 $\mu_1 = 100$, $\mu_2 = 100$, $cv = 0.1, 0.2, 0.3$

With this setting of demands, when the NV orders Q_1^*, store demand can probably be all satisfied in the selling season and no under-stock happens.

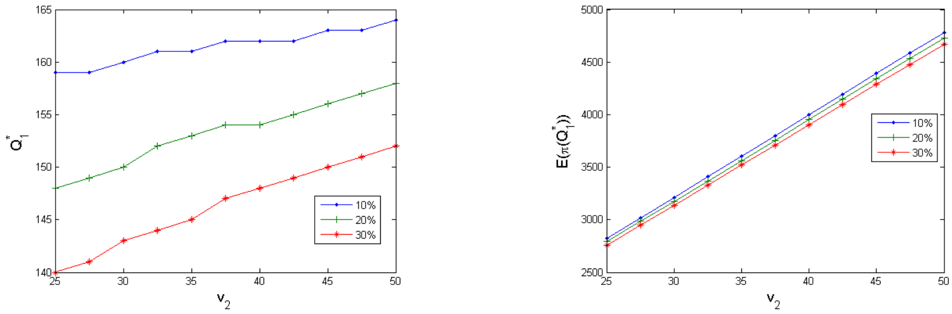
The unit store selling price has no impact on Q_1^* and $E(\pi(Q_1^*))$ increases with it, see Figure 4.12. When v_1 increases, obviously the NV wants more store sales. However, since all store demand are already satisfied by store inventory, increasing Q_1^* does not increase store sale. $E(\pi(Q_1^*))$ increases because the revenue related to each store sale increases with the unit selling price.

Q_1^* and $E(\pi(Q_1^*))$ both increase with the unit Internet selling price, see Figure 4.13. When unit Internet selling price is higher, the NV wants more Internet sale. The increase of optimal order quantity brings more Internet sale: when the order quantity is bigger, less Internet demand is lost because the return rate related to Internet demand satisfied by store inventory is smaller than by drop-shipping.

Numerical examples show that t and p have no impact on Q_1^* or $E(\pi(Q_1^*))$. As we explained, with this setting of demands, under-stock for store demand rarely happens, thus penalty and substitution are both negligible.

4.5.2.2 $\mu_1 = 100$, $\mu_2 = 10$, $cv = 0.1, 0.2, 0.3$

With this setting of demands, Q_1^* is much less than that with $\mu_2 = 100$, see Figures 4.14, 4.15. As a result, under-stock for store demand probably happens during the selling season.



(a) Optimal order quantity as a function of unit Internet selling price

(b) Optimal expected profit as a function of unit Internet selling price

Figure 4.13: impact of v_2

Q_1^* and $E(\pi(Q_1^*))$ increase with the unit store selling price, see Figure 4.14. The reason is that when v_1 increases, the NV wants more store sale. Since under-stock happens during the season, the NV can increase store sale by increasing Q_1^*. $E(\pi(Q_1^*))$ increases because the unit selling price increases.

Q_1^* decreases with the v_2 and $E(\pi(Q_1^*))$ increases, see Figure 4.15. When unit v_2 is higher, the NV reduce Q_1^* because unsatisfied store demand can be partly transferred to Internet demand payed by a higher unit selling price v_2. $E(\pi(Q_1^*))$ increases because the unit Internet selling price is higher.

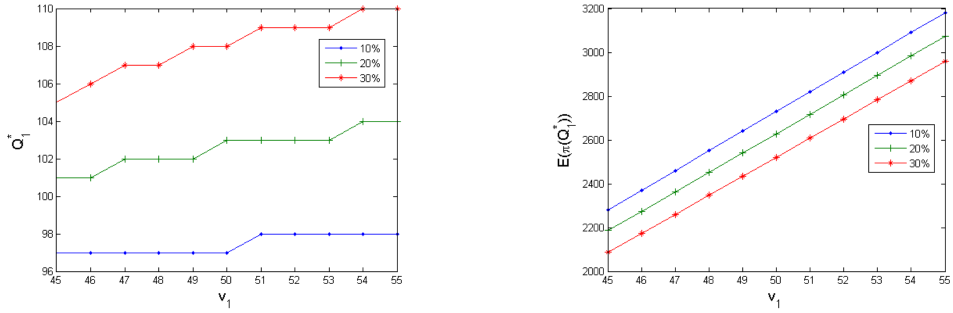
Q_1^* decreases with the substitution fraction t and $E(\pi(Q_1^*))$ increases, see Figure 4.16. Q_1^* decreases because when t is bigger, more unsatisfied store demand is substituted towards x_2, thus the NV can reduce Q_1^*. $E(\pi(Q_1^*))$ increases with t because more unsatisfied store demands are transferred to x_2 and satisfied by Q_2.

Q_1^* and $E(\pi(Q_1^*))$ both increase with the unit penalty cost, see Figure 4.17. The reason is that when p is bigger, unsatisfied store demand results higher penalty cost.

4.5.3 impact of ignoring product returns

If the NV ignores product returns when deciding the order quantity, he would not consider the part of the demand that is lost due to returns. When product returns are ignored, the optimal order quantity denoted as Q_1^0 is obtained by equation 4.19, the related expected profit is obtained by equation 6.42.

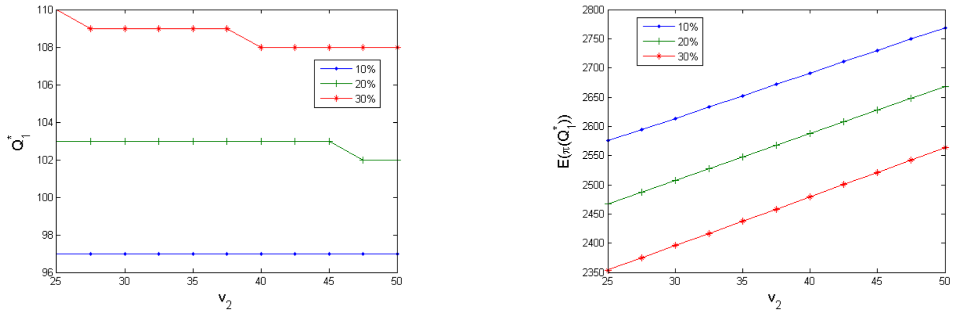
As expected, numerical examples show that $Q_1^0 > Q_1^*$ and $E(\pi(Q_1^0)) < E(\pi(Q_1^*))$.



(a) Optimal order quantity as a function of unit store selling price

(b) Optimal expected profit as a function of unit store selling price

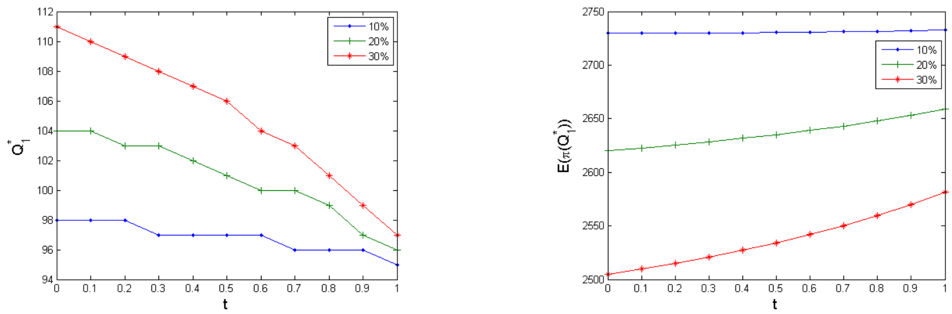
Figure 4.14: impact of v_1



(a) Optimal order quantity as a function of unit Internet selling price

(b) Optimal expected profit as a function of unit Internet store selling price

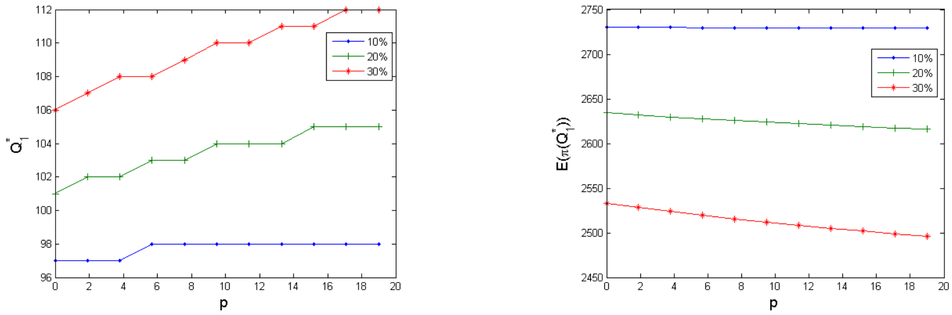
Figure 4.15: impact of v_2



(a) Optimal order quantity as a function of substitution fraction

(b) Optimal expected profit as a function of substitution fraction

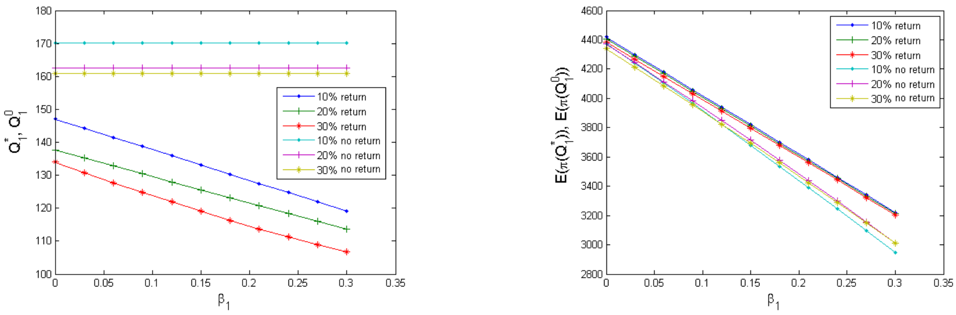
Figure 4.16: impact of t



(a) Optimal order quantity as a function of unit penalty cost

(b) Optimal expected profit as a function of unit penalty cost

Figure 4.17: impact of p



(a) Optimal order quantity as a function of store sale return rate

(b) Optimal expected profit as a function of store sale return rate

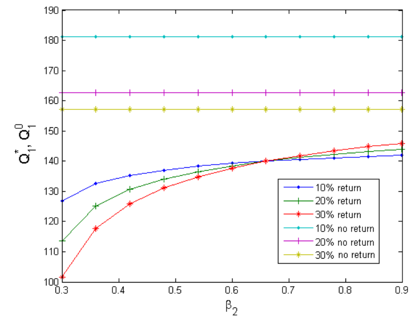
Figure 4.18: impact of return with $\mu_1 = 100$, $\mu_2 = 100$, $\beta_2 = \beta_3 = 0.3$

The difference between Q_1^0 and Q_1^*, so as to the difference between $E(\pi(Q_1^0))$ and $E(\pi(Q_1^*))$, increases with β_1 (see Figure 4.18), β_3 (see Figure 4.19), and decreases with β_2 (see Figure 4.20).

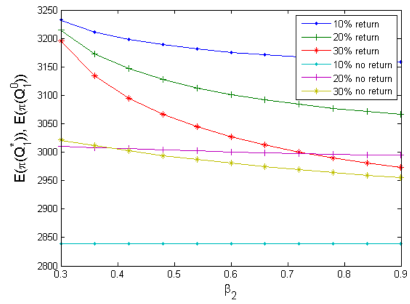
The worst case is when β_1 and β_3 are great, β_2 is small. For instance, in our numerical example, if we take $\beta_1 = \beta_2 = \beta_3 = 0.3$, ignoring product returns leads to an expected profit which is 14.4% less than the optimal expected profit.

4.5.4 impact of drop-shipping

If the NV does not use drop-shipping, the optimal order quantity denoted as Q_1^d is obtained by equation 4.14 by letting the first derivative equals zero. The related expected profit is obtained by doing the expect operation from equation 4.13.

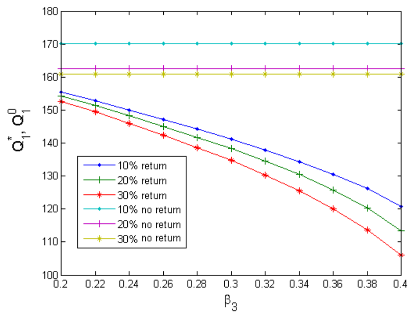


(a) Optimal order quantity as a function of store sale return rate

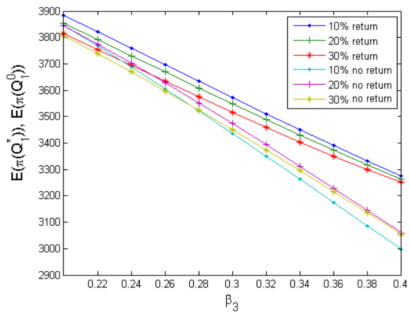


(b) Optimal expected profit as a function of store sale return rate

Figure 4.19: impact of return with $\mu_1 = 100$, $\mu_2 = 100$, $\beta_1 = \beta_3 = 0.3$

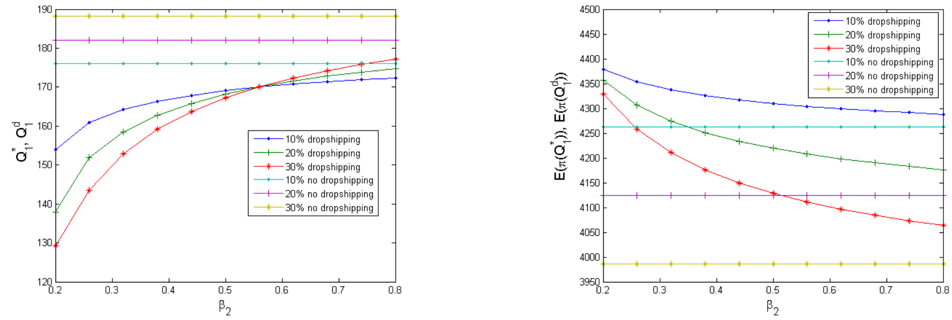


(a) Optimal order quantity as a function of store sale return rate



(b) Optimal expected profit as a function of store sale return rate

Figure 4.20: impact of return with $\mu_1 = 100$, $\mu_2 = 100$, $\beta_1 = 0.2$, $\beta_2 = 0.4$



(a) Optimal order quantity as a function of store sale return rate

(b) Optimal expected profit as a function of store sale return rate

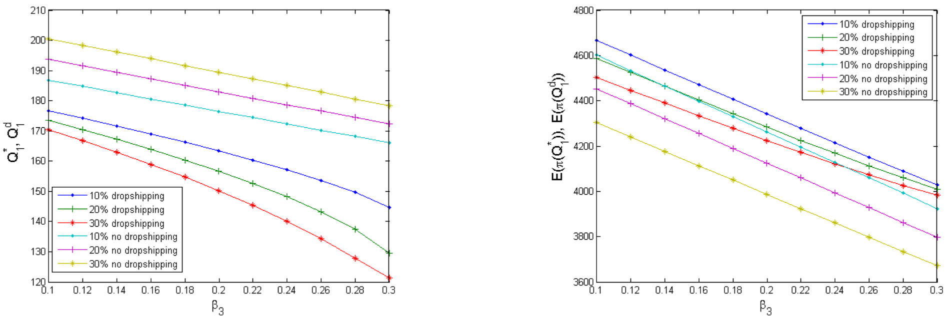
Figure 4.21: impact of drop-shipping with $\mu_1 = 100$, $\mu_2 = 100$, $\beta_1 = 0.1$, $\beta_3 = 0.2$

As expected, numerical examples show that $Q_1^d > Q_1^*$ $E(\pi(Q_1^d)) < E(\pi(Q_1^*))$. The reason is that NV can use drop-shipping for satisfying Internet demand. The difference between Q_1^d and Q_1^*, thus the difference between $E(\pi(Q_1^d))$ and $E(\pi(Q_1^*))$, decrease with β_2 (see Figure 4.21), because Q_1^* increases with β_2 and Q_1^d is constant. The difference increases with β_3 (see Figure 4.22) because when β_3 is bigger, the NV uses more drop-shipping for satisfying Internet demand for reducing Internet sale returns. The difference increases with μ_2 (see Figure 4.23) because when there is more Internet demand, there is more interest in using drop-shipping option. The difference does not have obvious change when β_1 increases, because the product returns related to store sale has no relation with drop-shipping option.

Without drop-shipping option, NV orders an inventory bigger than the optimal and the related expected profit will be less than the optimal. The worst case is when β_3 is big, β_2 is small, and μ_2 is big. In our example, if we take $\beta_1 = 0.1$, $\beta_2 = \beta_3 = 0.2$,$\mu_2 = 100$, $cv = 0.3$, using drop-shipping let the NV order a quantity 31.2% less and brings an expected profit 9.0% larger.

4.6 Conclusion

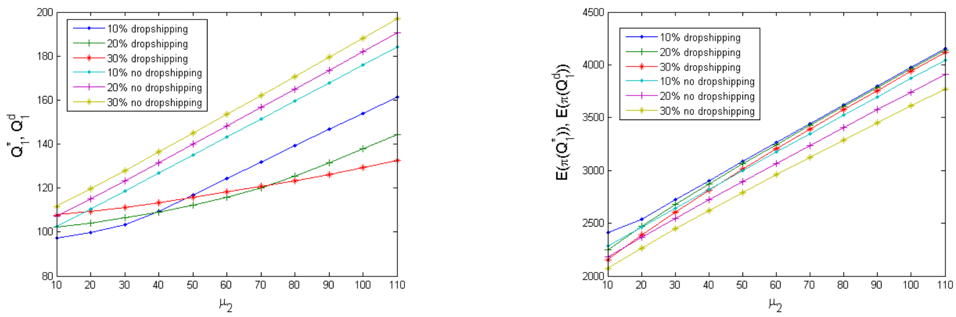
Using drop-shipping to satisfy demand is an interesting option for e-commerce retailers in order to reduce inventory related costs. However, the high product-return rate related to e-commerce business can challenge the use of drop-shipping as the only option for



(a) Optimal order quantity as a function of store sale return rate

(b) Optimal expected profit as a function of store sale return rate

Figure 4.22: impact of drop-shipping with $\mu_1 = 100$, $\mu_2 = 100$, $\beta_1 = 0.1$, $\beta_2 = 0.3$



(a) Optimal order quantity as a function of Internet mean demand

(b) Optimal expected profit as a function of Internet mean demand

Figure 4.23: impact of drop-shipping with $\mu_1 = 100$, $\beta_1 = 0.1$, $\beta_2 = 0.2$, $\beta_3 = 0.2$

satisfying demand. Many retailers prefer therefore to use a mixed drop-shipping and store inventory replenishment strategy to satisfy demand.

In this chapter we formulate a NV model for identifying the optimal mix of drop-shipping quantity and store inventory by considering different return rates for different types of flows (store inventory to store demand, drop-shipping to Internet demand and store inventory to Internet demand). We provide the optimal condition under general demand distributions as well as the optimal expected profit equation.

For two variants of Model, we demonstrate the concavity of the expected profit function and give the optimal expected profit equation. Some special cases are also considered: the case with no product returns, the case with identical product-return rates, the case with no store demand and the case with no drop-shipping option. Numerical analysis is provided to illustrate the impact of model parameters.

Our work presents also some limits. We have assumed that all returned products are resalable, while in practice this may not be the case. To solve this problem, a parameter of resalable products portion can be introduced. Such a parameter already exists in the NV model with returns but no drop-shipping option [87].

Another possible research can be to take the timing of returns into consideration: the single period can be extended to a multi-period problem where returned products arriving after the end of the first selling season can be resold at the next selling periods.

It will also be interesting to consider a problem where the suppliers (both drop-shipper and the distant supplier) offer quantity discounts in addition to the assumptions of our model. The mixed supplying strategy will be different according to the quantity discounts policies.

5

Conclusion and perspectives

Interest in the NVP has increased over the past 50 years. This interest can be attributed in part to the increased globalization. Also, the reduction in product life cycles brought about by technological advances makes the NVP more relevant. In this chapter, we give general concluding remarks and present directions for future research. For further details, we refer the reader to the concluding sections of the previous chapters.

In this thesis, we focused on three different extensions of the NVP: multiple discounts, product variety and free product returns policy. Our analysis leads to both qualitative and quantitative results.

In particular, we have investigated the impact of multiple discounts on inventory management. Using multiple discounts is a common way for retailers to deal with overstock in order to reduce the overage cost. This policy, in return, influences the optimal order quantity decision since the overage cost is reduced by using multiple discounts. We developed the model that provide the optimal ordering quantity for a NV using multiple discounts and showed insights on discount schemes. For instance, numerical results show that increasing discount numbers increases the expected profit: in our example, the expected profit is increased up to 100% with 5 progressive discounts compared with only one final discount. However, there is an upper limit of the expected profit when the NV increases the discount number.

Next, we analyzed the impact of product variety in inventory management. We developed a model considering the substitution effect and compared it with several models that can be used in practice. Considering the substitution and assortment effect significantly increases the expected profit up to 32% in our examples.

Then, we proposed a model for a NV that has a mixed supplying strategy (using both drop shipping and store inventory) and resalable returns related to store sale and Internet sale. We assumed that drop-shipping can be only used for Internet demand and store inventory can be used for both store demand and Internet demand. We investigated then the impact of the parameters with different demand settings, illustrated the impact of ignoring product returns and the benefit of drop-shipping. If the NV ignores the product returns, the expected profit is reduced by up to 14% compared with the optimal. The drop-shipping option, meanwhile, can bring an increase of expected profit up to 9%.

In the following, we provide some interesting research perspectives that can be developed as new extensions of the NVP.

One may extend our work by incorporating the effects of advertising in the NVP. Indeed, demand can be influenced not only by pricing but also by advertising. Many researchers assumed that the demand is a function of price, but few of them have taken into account the impact of advertising on demand. Advertising is a lever that is frequently used by companies to target consumers so that they buy more products. A joint determination of optimal order quantity and the advertising policy (e.g. the advertising spending) can be an interesting research area. Such kind of work is emerging recently in some papers, e.g. [91], but there are still lots of work that can be done considering the advertising effect in different situations, for instance, a NV often uses both pricing and advertising to influence the demand.

A second interesting research perspective is a multi-echelon NV structure, since in practice a NV can have a distribution center and some physical stores. For instance, when one develops a drop-shipping model used by the NV, internet demand can be satisfied by both the distribution center and the drop-shipper, while store demands only served by the stock in each store. Thus, the order quantity for distribution center inventory and the order quantity of drop-shipping need to be simultaneously considered.

Another possible area for future research lies on the consideration of a different objective for the NV. Earlier research is mainly based on the profit maximizing optimization objective, while sustainable supply chains is becoming a global need. Reducing overall carbon footprint, reducing energy and resource consumption can be considered while the NV optimizes the operations to achieve greater cost savings and profitability (e.g. [92]). The overage of stock, which brings wastes of energy and resource, for example, can be an interesting research area for sustainable NVP.

6

Appendices

6.1 Appendix of chapter 2

Appendix 1: Expected profit for additive price-dependent demand:

$$\begin{aligned}E(\pi(Q)) \;=\; & \int_{Q}^{\inf}[v_0Q - wQ]f(x)dx + \int_{\mu_0-\mu_1+Q}^{Q}[v_0x + (Q-x)v_1 - wQ]f(x)dx + \\ & + \int_{\mu_0-\mu_2+Q}^{\mu_0-\mu_1+Q}[v_0x + (\mu_1-\mu_0)v_1 - (\mu_1-\mu_0+x)v_2 + (v_2-w)Q]f(x)dx + \cdots \\ & + \int_{\mu_0-\mu_i+Q}^{\mu_0-\mu_{i-1}+Q}[v_0x + (\mu_1-\mu_0)v_1 + \cdots + (\mu_{i-1}-\mu_{i-2})v_{i-1} - (\mu_{i-1}-\mu_0 + \\ & + x)v_i + (v_i-w)Q]f(x)dx + \cdots + \int_{0}^{\mu_0-\mu_{n-1}+Q}[v_0x + (\mu_1-\mu_0)v_1 + \cdots + \\ & + (\mu_{n-1}-\mu_{n-2})v_{n-1} - (\mu_{n-1}-\mu_0+x)v_{n-1} + (v_n-w)Q]f(x)dx \qquad (6.1)\end{aligned}$$

$E(\pi(Q)) = Q[\int_Q^{\inf}(v_0 - w)f(x)dx + \int_0^{\mu_0-\mu_{i-1}+Q}(v_n - w) + f(x)dx + \int_{\mu_0-\mu_i+Q}^{\mu_0-\mu_{i-1}+Q}$

$(v_i - w)f(x)dx] + \int_{\mu_0-\mu_2+Q}^{\mu_0-\mu_1+Q}[v_0x + (\mu_1-\mu_0)v_1 - (\mu_1-\mu_0+x)v_2]f(x)dx + \cdots +$

$\int_{\mu_0-\mu_i+Q}^{\mu_0-\mu_{i-1}+Q}[v_0x + (\mu_1-\mu_0)v_1 + \cdots + (\mu_{i-1}-\mu_{i-2})v_{i-1} - (\mu_{i-1}-\mu_0+x)v_i]f(x)dx$

$+ \cdots + \int_0^{\mu_0-\mu_{n-1}+Q}[v_0x + (\mu_1-\mu_0)v_1 + \cdots + (\mu_{n-1}-\mu_{n-2})v_{n-1} - (\mu_{n-1}-\mu_0+x)v_{n-1}]f(x)dx$

$= Q[-w + v_0 + \sum_{i=0}^{n-1}(v_{i+1}-v_i)F(Q+\mu_0-\mu_i)] + \sum_{i=1}^{n-1}\sum_{j=0}^{i-1}\int_{\mu_0-\mu_i+Q}^{\mu_0-\mu_{i-1}+Q}(v_j -$

$v_{j+1})(x-\mu_j+\mu_0)f(x)dx + \sum_{j=0}^{n-1}\int_0^{\mu_0-\mu_{n-1}+Q}(v_j - v_{j+1})(x-\mu_j+\mu_0)f(x)dx$

$= Q[-w + v_0 + \sum_{i=0}^{n-1}(v_{i+1}-v_i)F(Q+\mu_0-\mu_i)] + \sum_{i=0}^{n-1}\int_0^{Q+u_0-u_i}(v_i - v_{i+1})(x+u_i-u_0)f(x)dx$

Appendix 2: Proof of lemma 1:

Use Leibniz's rule, we get the derivative of $E(Q)$:

$$\frac{dE(\pi(Q))}{dQ} = -\sum_{i=0}^{n-1}(v_i - v_{i+1})F(Q+\mu_0-\mu_i) + v_0 - w \qquad (6.2)$$

The second derivative of $E(\pi(Q))$ is:

$$\frac{d^2E(\pi(Q))}{d^2Q} = -\sum_{i=0}^{n-1}(v_i - v_{i+1})f(Q+\mu_0-\mu_i) \qquad (6.3)$$

$f(x) > 0$, $v_i - v_{i+1} > 0$, so $\frac{d^2E(\pi(Q))}{d^2Q} < 0$, then $E(\pi(Q))$ is concave.

Appendix 3: Expected profit of multiplicative price-dependent demand:

$$\begin{aligned}
E(\pi(Q)) &= \int_{Q}^{\inf}(v_0Q-wQ)f(x)dx+\int_{\frac{\mu_0}{\mu_1}Q}^{Q}(v_0x+(Q-x)v_1-wQ)f(x)dx+\\
&\int_{\frac{\mu_0}{\mu_2}Q}^{\frac{\mu_0}{\mu_1}Q}(v_0x+\frac{x}{\mu_0}((\mu_1-\mu_0)v_1-\mu_1v_2)+(v_2-w)Q)f(x)dx+\cdots\\
&+\int_{\frac{\mu_0}{\mu_i}Q}^{\frac{\mu_0}{\mu_{i-1}}Q}(v_0x+\frac{x}{\mu_0}((\mu_1-\mu_0)v_1+\cdots+\\
&+(\mu_{i-1}-\mu_{i-2})v_{i-1})-\frac{x_0v_i}{\mu 0}(\mu_{i-1})+(v_i-w)Q)f(x)dx+\cdots\\
&+\int_{0}^{\frac{\mu_0}{\mu_{n-1}}Q}(v_0x+\frac{x}{\mu_0}(\mu_1-\mu_0)v_1+\cdots+(\mu_{n-1}-\mu_{n-2})v_{n-1})-\frac{xv_n}{\mu_0}(\mu_{n-1})+\\
&+(v_n-w)Q)f(x)dx
\end{aligned} \tag{6.4}$$

Use Leibniz's rule, we get the derivative of $E(\pi(Q))$:

$$\frac{dE(\pi(Q))}{dQ}=-\sum_{i=0}^{n-1}(v_i-v_{i+1})F(Q\frac{\mu_0}{\mu_i})+v_0-w \tag{6.5}$$

Appendix 4: Proof of lemma 3:

The second derivative of $E(\pi(Q))$ is:

$$\frac{d^2E(\pi(Q))}{d^2Q}=-\sum_{i=0}^{n-1}(v_i-v_{i+1})f(Q\frac{\mu_0}{\mu_i})\frac{\mu_0}{\mu_i} \tag{6.6}$$

$f(x)>0$, $v_i-v_{i+1}>0$, so $\frac{d^2E(\pi(Q))}{d^2Q}<0$, then $E(\pi(Q))$ is concave.

Appendix 5: Numerical example for chapter 4.3

Consider the practical example: $w=3$, The amount of demand(x) has a normal distribution N(μ_0,σ_0) or U$[\mu_0-\sigma_0,\mu_0+\sigma_0]$, $\mu_0=a-bv, a=80, b=8, s=2$. According to lemma 2, $E(\pi(Q^*))-E_\sigma-E_v=\epsilon$, and $\epsilon=0$ in some conditions. In the linear discount case, according to equation2.11, the expected profit should be close to a hyperbola of n. And the maximum is:180. By setting $v_0=8$, we have $\mu_0=16$. Consider $\sigma_0=0$(deterministic demand),2,4,6,8, and n increases from 2. The expected profit $E(\pi(Q^*))$ is calculated by equation 2.5; Figure 6.1, 6.2 show the values of $E(\pi(Q^*))-E_\sigma$ and E_v.

It is obvious that ϵ increases with n, the reason is that when n is larger the condition of σ_0 tends to be not satisfied. In the uniform distribution case, the graph shows that

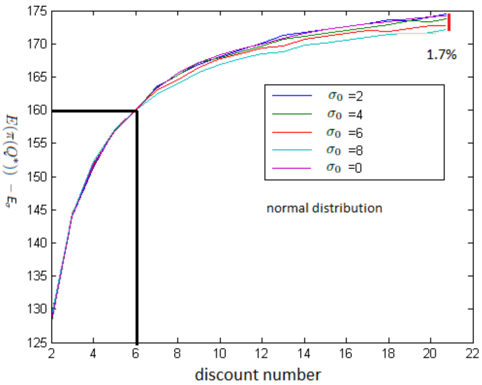


Figure 6.1: The value of $(E(\pi(Q^*)) - E_\sigma)$, as a function of discount number, with normal distribution

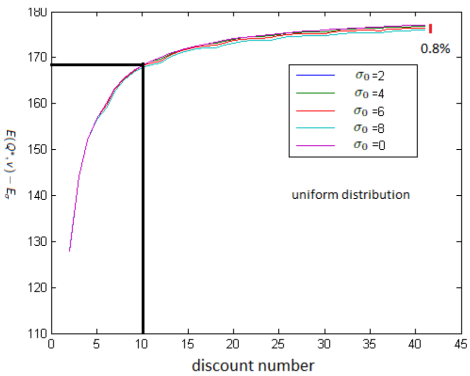


Figure 6.2: The value of $(E(\pi(Q^*)) - E_\sigma)$, as a function of discount number, with uniform distribution

$\epsilon = 0$ for $n < 11$; in the normal distribution demand case, it is exact for $n < 7$. In practice, this $n = 7$ is rather big for a multi-discount selling season. Even at $n = 21$, $\epsilon < 1.7\% E_v$. These results works for all $\sigma_0 \leq \mu/2$. Repeat the computation with different combinations w, s, a, b,v_0, similar results were got. Therefore, the error in equation ϵ is rather small or even zero.

6.2 Appendix of chapter 3

Appendix 1

$x_i^{'} = p_i^{'}x$, thus the probability that $x \in (a,b)$ equals to the probability that $x_i^{'} \in (p_i^{'}a, p_i^{'}b)$, regardless the value of a and b:

$$\begin{aligned} \int_{ap_i^{'}}^{bp_i^{'}} f_i^{'}(x_i^{'})dx_i^{'} &= \int_a^b f(x)dx \\ &= \int_{ap_i^{'}}^{bp_i^{'}} \frac{f(x)}{p_i^{'}} d(p_i^{'}x) \\ &= \int_{ap_i^{'}}^{bp_i^{'}} \frac{f(\frac{x_i^{'}}{p_i^{'}})}{p_i^{'}} dx_i^{'} \end{aligned} \tag{6.7}$$

This equation should be available for any value of a and b, thus we have equation 3.4.

Appendix 2

For policy 2, we have $F_i(q_i^*) = \frac{v_i - w_i}{v_i - s_i}$ and

$$E(\pi(Q^*)) = \sum_{i\in M}[\int_0^{q_i^*}(v_i - s_i)x_i f_i^{'}(x_i)dx_i - K_i] \tag{6.8}$$

For normal distribution,

$$\int_0^{q_i^*}(v_i - s_i)x_i f_i^{'}(x_i)dx_i = (v_i - w_i)\mu_i^{'} - (v_i - s_i)\sigma_i^{'}\sigma_i^{'}f_i^{'}(F_i^{'-1}(\frac{v_i - w_i}{v_i - s_i})) \tag{6.9}$$

When the products have same selling price, purchase cost and salvage value, $\sigma_i^{'}f_i^{'}(F_i^{'-1}(\frac{v-w}{v-s})) = f_0(F_0^{-1}(\frac{v-w}{v-s}))$ is a constant value, thus

$$E(\pi(Q^*)) = \sum_{i\in M}(\mu_i A + \sigma_i B) = \sum_{i\in M} p_i^{'}(\mu A + \sigma B) \tag{6.10}$$

With $A = v - w, B = -(v-s)f_0(F_0^{-1}(\frac{v-w}{v-s})), f_0$ is the standard normal probability density function and F_0 is the standard normal cumulative distribution function.

For uniform distribution $[\mu - \sigma, \mu + \sigma]$,

$$\int_0^{q_i^*}(v_i - s_i)x_i f_i^{'}(x_i)dx_i = (v_i - w_i)\mu_i^{'} - (v_i - s_i)\sigma_i^{'}\frac{1 - (2\frac{v_i - w_i}{v_i - s} - 1)^2}{4} \tag{6.11}$$

When the products have same selling price, purchase cost and salvage value,

$$E(\pi(Q^*)) = \sum_{i \in M} (\mu_i A + \sigma_i B) = \sum_{i \in M} p_i^{'}(\mu A + \sigma B) \tag{6.12}$$

With $A = v - w, B = -\frac{1-(2\frac{v-w}{v-s}-1)^2}{4}$.

For exponential distribution with parameter λ_i,we have$\mu_i = 1/\lambda_i, \sigma_i = 1/\lambda_i$. We have

$$\begin{aligned}
\int_0^{q_i^*} (v_i - s_i) x_i f_i^{'}(x_i) dx_i &= (v_i - s_i)(-q_i^* e^{-\lambda_i q_i^*} - \frac{e^{-\lambda_i q_i^*} - 1}{\lambda_i}) \\
&= \frac{v_i - w_i}{\lambda_i} - \frac{(w_i - s_i)\ln\frac{v_i - s_i}{w_i - s_i}}{\lambda_i} \\
&= (v_i - w_i)\mu_i - (w_i - s_i)\sigma_i \ln\frac{v_i - s_i}{w_i - s_i}
\end{aligned} \tag{6.13}$$

When the products have same selling price, purchase cost and salvage value,

$$E(\pi(Q^*)) = \sum_{i \in M} (\mu_i A + \sigma_i B) = \sum_{i \in M} p_i^{'}(\mu A + \sigma B) \tag{6.14}$$

With $A = (v - w), B = -(w - s)\ln\frac{v-s}{w-s}$.

Appendix 3: combinations of (K, L, σ) that maximize the difference between policy 4 and 5

In this part, we are interested in values of σ, K and L that maximize the difference between policy 4 and 5. As shown in Figure 3.10 and 3.8, policy 4 get the same results as policy 5, except for a few cases. The analysis of these cases shows that there are some points maximizing the difference between policy 4 and 5. At these points, policy 2 shows an interesting character: the expected profit calculated by policy 2 E_m corresponding to the optimal assortment M including m product variants is very close to the expected profit E_{m+1} related to the assortment with an additional product j, i.e. $E_m \approx E_{m+1}$. $E_m = \sum_{i \in M}(\mu_i A + \sigma_i B) = \sum_{i \in M} p_i^{'}(\mu A + \sigma B)$. With normally distributed demand, for example, $A = v - w, B = -(v - s)f_0(F_0(\frac{v-w}{v-s})), f_0$ is the standard normal distribution density function. A necessary condition for maximizing the difference between policy 4 and 5 is derived:

$$p_j^{'}(\mu A + \sigma B) = \frac{K}{L} \tag{6.15}$$

Take Figure 3.10 for example, $L = 0.3$, $\sigma = 20$ thus $\mu A + \sigma B = 240$. The difference between policy 4 and 5 gets maximum when the optimal assortment size m equals 1.

Thus the additional product j is the product with the second largest market share: $p_j = 0.25$. According the equation 6.15, the fixed display cost K that maximize the difference is: $K = p_j^{'}(\mu A + \sigma B)L = 18$. Which is close to the result seen in Figure 3.10: $K = 20$. It is not exactly the same value to the one derived by equation 6.15 because the case $K = 18$ is not shown in the figure.

6.3 Appendix of chapter 4

Appendix 1: Proposition 1

Proof. Let

$$\alpha_1 = v_1(1-\beta_1) - w_r\beta_1 - s(1-\beta_1) \tag{6.16}$$

$$b_1 = v_2(1-\beta_3) - w_r\beta_3 - s(1-\beta_3) \tag{6.17}$$

$$\lambda_1 = -w_1 + s \tag{6.18}$$

$$\alpha_2 = v_1(1-\beta_1) - v_2(1-\beta_1) + \frac{1-\beta_1}{1-\beta_3}\beta_3 w_r - \beta_1 w_r - (w_2 + \tfrac{\beta_2 w_r}{1-\beta_3} - v_2)\tfrac{1-\beta_1}{1-\beta_3+\beta_2} \tag{6.19}$$

$$b_2 = (v_2 - w_2 - \frac{\beta_2 w_r}{1-\beta_3})\frac{1-\beta_3}{1-\beta_3+\beta_2} \tag{6.20}$$

$$\lambda_2 = v_2 - w_1 - \frac{\beta_3 w_r}{1-\beta_3} + (w_2 + \frac{\beta_2 w_r}{1-\beta_3} - v_2)\frac{1}{1-\beta_3+\beta_2} \tag{6.21}$$

$$\alpha_3 = \frac{(1-\beta_3)(v_2-w_2)t}{1-\beta_3+\beta_2} - \frac{\beta_2 w_r t}{1-\beta_3+\beta_2} - (1-t)p \tag{6.22}$$

$$b_3 = (v_2 - w_2 - \frac{\beta_2 w_r}{1-\beta_3})\frac{1-\beta_3}{1-\beta_3+\beta_2} \tag{6.23}$$

$$\lambda_3 = v_1 - w_1 - \frac{\beta_1 w_r}{1-\beta_1} - \frac{1}{1-\beta_1}(\frac{(1-\beta_3)(v_2-w_2)t}{1-\beta_3+\beta_2} - \frac{\beta_2 w_r t}{1-\beta_3+\beta_2}) + \frac{(1-t)p}{1-\beta_1} \tag{6.24}$$

Substituting them in equation 4.2 gives

$$\pi = \begin{cases} \alpha_1 x_1 + b_1 x_2 + \lambda_1 Q_1 & \text{case 1} \\ \alpha_2 x_1 + b_2 x_2 + \lambda_2 Q_1 & \text{case 2} \\ \alpha_3 x_1 + b_3 x_2 + \lambda_3 Q_1 & \text{case 3} \end{cases} \tag{6.25}$$

Now we search the optimal order quantity which maximizes the expected profit, $E(\pi)$.

$$\begin{aligned} E(\pi) = &\int_{-\infty}^{\frac{Q_1}{1-\beta_1}} [\int_{-\infty}^{\frac{Q_1}{1-\beta_3} - \frac{1-\beta_1}{1-\beta_3}x_1} (\alpha_1 x_1 + b_1 x_2 + \lambda_1 Q_1) f_2(x_2) dx_2 \\ &+ \int_{\frac{Q_1}{1-\beta_3} - \frac{1-\beta_1}{1-\beta_3}x_1}^{\infty} (\alpha_2 x_1 + b_2 x_2 + \lambda_2 Q_1) f_2(x_2) dx_2] f_1(x_1) dx_1 \\ &+ \int_{\frac{Q_1}{1-\beta_1}}^{\infty} [\int_{-\infty}^{\infty} (\alpha_3 x_1 + b_3 x_2 + \lambda_3 Q_1) f_2(x_2) dx_2] f_1(x_1) dx_1 \end{aligned} \tag{6.26}$$

The first derivative is derived as:

$$\begin{aligned}\frac{dE(\pi)}{dQ_1} &= \lambda_2 F_1(\frac{Q_1}{1-\beta_1}) + \lambda_3(1 - F_1(\frac{Q_1}{1-\beta_1})) \\ &+ \int_{-\infty}^{\frac{Q_1}{1-\beta_1}} (\lambda_1 - \lambda_2) F_2(\frac{Q_1}{1-\beta_3} - \frac{1-\beta_1}{1-\beta_3}x_1) f_1(x_1) dx_1\end{aligned} \tag{6.27}$$

$$\begin{aligned}\frac{d^2E(\pi)}{dQ_1^2} &= -(\lambda_3 - \lambda_2) f_1(\frac{Q_1}{1-\beta_1}) \\ &- (\lambda_2 - \lambda_1) \int_{-\infty}^{\frac{Q_1}{1-\beta_1}} f_2(\frac{Q_1}{1-\beta_3} - \frac{1-\beta_1}{1-\beta_3}x_1) f_1(x_1) dx_1/(1-\beta_3)\end{aligned} \tag{6.28}$$

We have $\lambda_3 - \lambda_2 = v_1 - v_2 + \frac{w_r\beta_3}{1-\beta_3} - \frac{w_r\beta_1}{1-\beta_1} - \frac{w_2 + \frac{\beta_2 w_r}{1-\beta_3} - v_2}{1-\beta_3+\beta_2} - \frac{1}{1-\beta_1}(\frac{(1-\beta_3)(v_2-w_2)t}{1-\beta_3+\beta_2} - \frac{\beta_2 w_r t}{1-\beta_3+\beta_2} - (1-t)p)$, it is easy to find that

$$\frac{w_r\beta_3}{1-\beta_3} - \frac{w_r\beta_1}{1-\beta_1} > 0$$

And

$$\begin{aligned}& v_1 - v_2 - \frac{w_2 + \frac{\beta_2 w_r}{1-\beta_3} - v_2}{1-\beta_3+\beta_2} - \frac{1}{1-\beta_1}(\frac{(1-\beta_3)(v_2-w_2)t}{1-\beta_3+\beta_2} - \frac{\beta_2 w_r t}{1-\beta_3+\beta_2} - (1-t)p) \\ &= \frac{v_1 + v_1(\beta_2-\beta_3) - w_2 - \frac{\beta_2 w_r}{1-\beta_3} + v_2(\beta_3-\beta_2)}{1-\beta_3+\beta_2} - \frac{1}{1-\beta_1}(\frac{(1-\beta_3)(v_2-w_2)t}{1-\beta_3+\beta_2} - \frac{\beta_2 w_r t}{1-\beta_3+\beta_2} - (1-t)p) \\ &= \frac{v_1 + (v_1-v_2)(\beta_2-\beta_3) - w_2 - \frac{\beta_2 w_r}{1-\beta_3}}{1-\beta_3+\beta_2} - \frac{1}{1-\beta_1}(\frac{(1-\beta_3)(v_2-w_2)t}{1-\beta_3+\beta_2} - \frac{\beta_2 w_r t}{1-\beta_3+\beta_2} - (1-t)p) \\ &\geq \frac{v_1 - (v_1-v_2) - w_2 - \frac{\beta_2 w_r}{1-\beta_3}}{1-\beta_3+\beta_2} - \frac{1}{1-\beta_1}(\frac{(1-\beta_3)(v_2-w_2)t}{1-\beta_3+\beta_2} - \frac{\beta_2 w_r t}{1-\beta_3+\beta_2} - (1-t)p) \\ &= \frac{v_2 - w_2 - \frac{\beta_2 w_r}{1-\beta_3}}{1-\beta_3+\beta_2} - \frac{1}{1-\beta_1}(\frac{(1-\beta_3)(v_2-w_2)t}{1-\beta_3+\beta_2} - \frac{\beta_2 w_r t}{1-\beta_3+\beta_2} - (1-t)p) \\ &= \frac{(1-t)p}{1-\beta_1} + \frac{v_2 - w_2 - \frac{\beta_2 w_r}{1-\beta_3}}{1-\beta_3+\beta_2} \frac{(1-\beta_1) - t(1-\beta_3)}{1-\beta_1} > 0\end{aligned}$$

Thus

$$\lambda_3 - \lambda_2 > 0$$

When $\beta_2 \geq \beta_3 - \frac{w_2 - s}{v_2 + w_r - s}$, we have

$$\lambda_2 - \lambda_1 = \frac{w_2 + (v_2 + w_r)(\beta_2 - \beta_3)}{1-\beta_3+\beta_2} - s > 0$$

The probability functions are non-negative, thus $\frac{d^2E(\pi)}{dQ_1^2} < 0$. □

Appendix 2: The optimal expected profit

The proof is similar as in Appendix 4.

Appendix 3: Proposition 3

Proof. Let

$$\alpha_1 = v_1(1-\beta_1) - w_r\beta_1 - s(1-\beta_1) \tag{6.29}$$

$$b_1 = v_2(1-\beta_3) - w_r\beta_3 - s(1-\beta_3) \tag{6.30}$$

$$\lambda_1 = -w_1 + s \tag{6.31}$$

$$\alpha_2 = v_1(1-\beta_1) - v_2(1-\beta_1) + \frac{1-\beta_1}{1-\beta_3}\beta_3 w_r - \beta_1 w_r - (w_2 + \tfrac{\beta_2 w_r}{1-\beta_3} - v_2)\tfrac{1-\beta_1}{1-\beta_3+\beta_2} \tag{6.32}$$

$$b_2 = (v_2 - w_2 - \frac{\beta_2 w_r}{1-\beta_3})\frac{1-\beta_3}{1-\beta_3+\beta_2} \tag{6.33}$$

$$\lambda_2 = v_2 - w_1 - \frac{\beta_3 w_r}{1-\beta_3} + (w_2 + \frac{\beta_2 w_r}{1-\beta_3} - v_2)\frac{1}{1-\beta_3+\beta_2} \tag{6.34}$$

$$\alpha_3 = v_1(1-\beta_1) - v_2(1-\beta_1) + \frac{1-\beta_1}{1-\beta_3}\beta_3 w_r - \beta_1 w_r - (w_2 + \frac{\beta_2 w_r}{1-\beta_3} - v_2)\frac{1-\beta_1}{1-\beta_3+\beta_2} \tag{6.35}$$

$$b_3 = (v_2 - w_2 - \frac{\beta_2 w_r}{1-\beta_3})\frac{1-\beta_3}{1-\beta_3+\beta_2} \tag{6.36}$$

$$\lambda_3 = v_2 - w_1 - \frac{\beta_3 w_r}{1-\beta_3} + (w_2 + \frac{\beta_2 w_r}{1-\beta_3} - v_2)\frac{1}{1-\beta_3+\beta_2} \tag{6.37}$$

$$\alpha_4 = \frac{(1-\beta_3)(v_2-w_2)t}{1-\beta_3+\beta_2} - \frac{\beta_2 w_r t}{1-\beta_3+\beta_2} - (1-t)p \tag{6.38}$$

$$b_4 = v_2(1-\beta_2) - (w_2 + \frac{\beta_2 w_r}{1-\beta_1} - v_1\beta_2) - \frac{\beta_2}{1-\beta_1}(\frac{(1-\beta_3)(v_2-w_2)t}{1-\beta_3+\beta_2} - \frac{\beta_2 w_r t}{1-\beta_3+\beta_2} - (1-t)p) \tag{6.39}$$

$$\lambda_4 = -w_1 - \frac{\beta_1 w_r}{1-\beta_1} + v_1 - \frac{1}{1-\beta_1}(\frac{(1-\beta_3)(v_2-w_2)t}{1-\beta_3+\beta_2} - \frac{\beta_2 w_r t}{1-\beta_3+\beta_2} - (1-t)p) \tag{6.40}$$

Substituting them in equation 4.16 gives

$$\pi = \begin{cases} \alpha_1 x_1 + b_1 x_2 + \lambda_1 Q_1 & \text{case 1} \\ \alpha_2 x_1 + b_2 x_2 + \lambda_2 Q_1 & \text{case 2} \\ \alpha_3 x_1 + b_3 x_2 + \lambda_3 Q_1 & \text{case 3} \\ \alpha_4 x_1 + b_4 x_2 + \lambda_4 Q_1 & \text{case 4} \end{cases} \tag{6.41}$$

Now we search the optimal order quantity which maximizes the expected profit, $E(\pi)$.

$$E(\pi) = \int_{-\infty}^{\frac{Q_1}{1-\beta_1}} [\int_{-\infty}^{\frac{Q_1}{1-\beta_3}-\frac{1-\beta_1}{1-\beta_3}x_1} (\alpha_1 x_1 + b_1 x_2 + \lambda_1 Q_1) f_2(x_2) dx_2$$

$$+ \int_{\frac{Q_1}{1-\beta_3}-\frac{1-\beta_1}{1-\beta_3}x_1}^{\infty} (\alpha_2 x_1 + b_2 x_2 + \lambda_2 Q_1) f_2(x_2) dx_2] f_1(x_1) dx_1 \tag{6.42}$$

$$+ \int_{\frac{Q_1}{1-\beta_1}}^{\infty} [\int_{\frac{x_1(1-\beta_1)-Q_1}{\beta_2}}^{\infty} (\alpha_3 x_1 + b_3 x_2 + \lambda_3 Q_1) f_2(x_2) dx_2$$

$$+ \int_{-\infty}^{\frac{x_1(1-\beta_1)-Q_1}{\beta_2}} (\alpha_4 x_1 + b_4 x_2 + \lambda_4 Q_1) f_2(x_2) dx_2] f_1(x_1) dx_1$$

The first derivative is derived as (c.f. Appendix 1):

$$\frac{dE(\pi)}{dQ_1} = \lambda_3 + \int_{\frac{Q_1}{1-\beta_1}}^{\infty} (\lambda_4 - \lambda_3) F_2(\frac{x_1(1-\beta_1)-Q_1}{\beta_2}) f_1(x_1) dx_1 \tag{6.43}$$

$$+ \int_{-\infty}^{\frac{Q_1}{1-\beta_1}} (\lambda_1 - \lambda_2) F_2(\frac{Q_1}{1-\beta_3} - \frac{1-\beta_1}{1-\beta_3} x_1) f_1(x_1) dx_1$$

Use Leibnizs rule, the first derivative of equation 6.42 is:

$$\begin{aligned}
\frac{dE(\pi)}{dQ_1} = \Bigg\{ \int_{-\infty}^{\frac{Q_1}{1-\beta_1}} \Bigg\{ \int_{\frac{Q_1}{1-\beta_3}-\frac{1-\beta_1}{1-\beta_3}x_1}^{\infty} (\lambda_2 f_2(x_2)dx_2 \\
- \frac{1}{1-\beta_3}(\alpha_2 x_1 + b_2(\frac{Q_1}{1-\beta_3} - \frac{1-\beta_1}{1-\beta_3}x_1) + \lambda_2 Q_1) f_2(\frac{Q_1}{1-\beta_3} - \frac{1-\beta_1}{1-\beta_3}x_1)) \\
+ \int_{-\infty}^{\frac{Q_1}{1-\beta_3}-\frac{1-\beta_1}{1-\beta_3}x_1} (\lambda_1 f_2(x_2)dx_2 + \frac{1}{1-\beta_3}(\alpha_1 x_1 + b_1(\frac{Q_1}{1-\beta_3} - \frac{1-\beta_1}{1-\beta_3}x_1) \\
+ \lambda_1 Q_1) f_2(\frac{Q_1}{1-\beta_3} - \frac{1-\beta_1}{1-\beta_3}x_1) \Bigg\} f_1(x_1)dx_1 \\
+ \frac{1}{1-\beta_1}\int_0^{\infty}(\alpha_2 \frac{Q_1}{1-\beta_1} + b_2 x_2 + \lambda_2 Q_1) f_1(\frac{Q_1}{1-\beta_1}) \Bigg\} \\
+ \Bigg\{ \int_{\frac{Q_1}{1-\beta_1}}^{\infty} \Bigg\{ \int_{\frac{x_1(1-\beta_1)-Q_1}{\beta_2}}^{\infty} \lambda_3 f_2(x_2)dx_2 \\
+ \frac{1}{\beta_2}(\alpha_3 x_1 + b_3 \frac{x_1(1-\beta_1)-Q_1}{\beta_2} + \lambda_3 Q_1) f_2(\frac{x_1(1-\beta_1)-Q_1}{\beta_2}) \\
+ \int_{-\infty}^{\frac{x_1(1-\beta_1)-Q_1}{\beta_2}} (\lambda_4 f_2(x_2)dx_2 \\
- \frac{1}{\beta_2}(\alpha_4 x_1 + b_4 \frac{x_1(1-beta_1)-Q_1}{\beta_2} + \lambda_4 Q_1) f_2(\frac{x_1(1-\beta_1)-Q_1}{\beta_2}) \Bigg\} f_1(x_1)dx_1 \\
- \frac{1}{1-\beta_1}\int_0^{\infty}(\alpha_3 \frac{Q_1}{1-\beta_1} + b_3 x_2 + \lambda_3 Q_1) f_1(\frac{Q_1}{1-\beta_1}) \Bigg\}
\end{aligned} \tag{6.44}$$

Because $\alpha_3 = \alpha_2, b_3 = b_2, \lambda_3 = \lambda_2$,

$$\begin{aligned}
-\frac{1}{1-\beta_1}\int_0^{\infty}(\alpha_3 \frac{Q_1}{1-\beta_1} + b_3 x_2 + \lambda_3 Q_1) f_1(\frac{Q_1}{1-\beta_1}) \\
+ \frac{1}{1-\beta_1}\int_0^{\infty}(\alpha_2 \frac{Q_1}{1-\beta_1} + b_2 x_2 + \lambda_2 Q_1) f_1(\frac{Q_1}{1-\beta_1}) = 0
\end{aligned} \tag{6.45}$$

$$\begin{aligned}
\frac{1}{\beta_2}(\alpha_3 x_1 + b_3 \frac{x_1(1-\beta_1)-Q_1}{\beta_2} + \lambda_3 Q_1) f_2(\frac{x_1(1-\beta_1)-Q_1}{\beta_2}) \\
- \frac{1}{\beta_2}(\alpha_4 x_1 + b_4 \frac{x_1(1-\beta_1)-Q_1}{\beta_2} + \lambda_4 Q_1) f_2(\frac{x_1(1-\beta_1)-Q_1}{\beta_2}) = \frac{1}{\beta_2}((\alpha_3-\alpha_4)x_1 \\
+ (b_3-b_4)\frac{x_1(1-\beta_1)-Q_1}{\beta_2} + (\lambda_3-\lambda_4)Q_1) f_2(\frac{x_1(1-\beta_1)-Q_1}{\beta_2})
\end{aligned} \tag{6.46}$$

Because $\alpha_3 - \alpha_4 = -(1-\beta_1)(\lambda_3 - \lambda_4) = -\frac{1-\beta_1}{\beta_2}(b_3 - b_4)$, equation 6.46 equals to 0. We can prove in the same way that

$$\begin{aligned}
&-\frac{1}{1-\beta_3}(\alpha_2 x_1 + b_2(\frac{Q_1}{1-\beta_3} - \frac{1-\beta_1}{1-\beta_3}x_1) + \lambda_2 Q_1) f_2(\frac{Q_1}{1-\beta_1} - x_1) + \\
&\frac{1}{1-\beta_3}(\alpha_1 x_1 + b_1(\frac{Q_1}{1-\beta_3} - \frac{1-\beta_1}{1-\beta_3}x_1) + \lambda_1 Q_1) f_2(\frac{Q_1}{1-\beta_1} - x_1) = 0
\end{aligned} \tag{6.47}$$

The first derivative can then be derived

$$\begin{aligned}
\frac{dE(\pi)}{dQ_1} &= \int_{-\infty}^{\frac{Q_1}{1-\beta_1}} \left\{ \int_{\frac{Q_1}{1-\beta_3} - \frac{1-\beta_1}{1-\beta_3}x_1}^{\infty} (\lambda_2 f_2(x_2) dx_2 + \int_{-\infty}^{\frac{Q_1}{1-\beta_3} - \frac{1-\beta_1}{1-\beta_3}x_1} (\lambda_1 f_2(x_2) dx_2 \right\} \\
&\quad + \int_{\frac{Q_1}{1-\beta_1}}^{\infty} \left\{ \int_{\frac{x_1(1-\beta_1)-Q_1}{\beta_2}}^{\infty} \lambda_3 f_2(x_2) dx_2 + \int_{-\infty}^{\frac{x_1(1-\beta_1)-Q_1}{\beta_2}} (\lambda_4 f_2(x_2) dx_2 \right\} \\
&= \int_{-\infty}^{\frac{Q_1}{1-\beta_1}} \left\{ \lambda_2 (1 - F_2(\frac{Q_1}{1-\beta_3} - \frac{1-\beta_1}{1-\beta_3}x_1)) \right. \\
&\left. + \lambda_1 F_2(\frac{Q_1}{1-\beta_3} - \frac{1-\beta_1}{1-\beta_3}x_1) \right\} + \int_{\frac{Q_1}{1-\beta_1}}^{\infty} \left\{ \lambda_3 (1 - F_2(\frac{x_1(1-\beta_1) - Q_1}{\beta_2})) \right. \\
&\left. + \lambda_4 F_2(\frac{x_1(1-\beta_1) - Q_1}{\beta_2}) \right\} \\
&= \lambda_3 + \int_{\frac{Q_1}{1-\beta_1}}^{\infty} (\lambda_4 - \lambda_3) F_2(\frac{x_1(1-\beta_1) - Q_1}{\beta_2}) f_1(x_1) dx_1 \\
&+ \int_{-\infty}^{\frac{Q_1}{1-\beta_1}} (\lambda_1 - \lambda_2) F_2(\frac{Q_1}{1-\beta_3} - \frac{1-\beta_1}{1-\beta_3}x_1) f_1(x_1) dx_1
\end{aligned} \tag{6.48}$$

$$\begin{aligned}
\frac{d^2 E(\pi)}{dQ_1^2} &= -(\lambda_4 - \lambda_3) \int_{\frac{Q_1}{1-\beta_1}}^{\infty} f_2(\frac{x_1(1-\beta_1) - Q_1}{\beta_2}) f_1(x_1) dx_1 / \beta_2 \\
&- (\lambda_2 - \lambda_1) \int_{-\infty}^{\frac{Q_1}{1-\beta_1}} f_2(\frac{Q_1}{1-\beta_3} - \frac{1-\beta_1}{1-\beta_3}x_1) f_1(x_1) dx_1 / (1-\beta_3)
\end{aligned} \tag{6.49}$$

We have $\lambda_4 - \lambda_3 = v_1 - v_2 + \frac{w_r\beta_3}{1-\beta_3} - \frac{w_r\beta_1}{1-\beta_1} - \frac{w_2 + \frac{\beta_2 w_r}{1-\beta_3} - v_2}{1-\beta_3+\beta_2} - \frac{1}{1-\beta_1}(\frac{(1-\beta_3)(v_2-w_2)t}{1-\beta_3+\beta_2} - \frac{\beta_2 w_r t}{1-\beta_3+\beta_2} - (1-t)p)$. It is easy to find that

$$\frac{w_r\beta_3}{1-\beta_3} - \frac{w_r\beta_1}{1-\beta_1} > 0$$

And

$$v_1 - v_2 - \frac{w_2 + \frac{\beta_2 w_r}{1-\beta_3} - v_2}{1-\beta_3+\beta_2} - \frac{1}{1-\beta_1}(\frac{(1-\beta_3)(v_2-w_2)t}{1-\beta_3+\beta_2} - \frac{\beta_2 w_r t}{1-\beta_3+\beta_2} - (1-t)p)$$

$$= \frac{v_1 + v_1(\beta_2 - \beta_3) - w_2 - \frac{\beta_2 w_r}{1-\beta_3} + v_2(\beta_3 - \beta_2)}{1 - \beta_3 + \beta_2} - \frac{1}{1-\beta_1}(\frac{(1-\beta_3)(v_2 - w_2)t}{1-\beta_3+\beta_2} - \frac{\beta_2 w_r t}{1-\beta_3+\beta_2} - (1-t)p)$$

$$= \frac{v_1 + (v_1 - v_2)(\beta_2 - \beta_3) - w_2 - \frac{\beta_2 w_r}{1-\beta_3}}{1 - \beta_3 + \beta_2} - \frac{1}{1-\beta_1}(\frac{(1-\beta_3)(v_2 - w_2)t}{1-\beta_3+\beta_2} - \frac{\beta_2 w_r t}{1-\beta_3+\beta_2} - (1-t)p)$$

$$\geq \frac{v_1 - (v_1 - v_2) - w_2 - \frac{\beta_2 w_r}{1-\beta_3}}{1 - \beta_3 + \beta_2} - \frac{1}{1-\beta_1}(\frac{(1-\beta_3)(v_2 - w_2)t}{1-\beta_3+\beta_2} - \frac{\beta_2 w_r t}{1-\beta_3+\beta_2} - (1-t)p)$$

$$= \frac{v_2 - w_2 - \frac{\beta_2 w_r}{1-\beta_3}}{1 - \beta_3 + \beta_2} - \frac{1}{1-\beta_1}(\frac{(1-\beta_3)(v_2 - w_2)t}{1-\beta_3+\beta_2} - \frac{\beta_2 w_r t}{1-\beta_3+\beta_2} - (1-t)p)$$

$$= \frac{(1-t)p}{1-\beta_1} + \frac{v_2 - w_2 - \frac{\beta_2 w_r}{1-\beta_3}}{1 - \beta_3 + \beta_2} \frac{(1-\beta_1) - t(1-\beta_3)}{1-\beta_1} > 0$$

Thus

$$\lambda_4 - \lambda_3 > 0$$

When $\beta_2 \geq \beta_3 - \frac{w_2 - s}{v_2 + w_r - s}$, we have

$$\lambda_2 - \lambda_1 = \frac{w_2 + (v_2 + w_r)(\beta_2 - \beta_3)}{1 - \beta_3 + \beta_2} - s > 0$$

The probability functions are non-negative, thus $\frac{d^2 E(\pi)}{dQ_1^2} < 0$. □

Appendix 4: The optimal expected profit

We develop equation 6.42 as the sum of three parts: x_1, x_2 and Q_1:

$$
\begin{aligned}
E(\pi) = \Big\{ & \int_{-\infty}^{\frac{Q_1}{1-\beta_1}} [\alpha_1 x_1 F_2(\frac{Q_1}{1-\beta_3} - \frac{1-\beta_1}{1-\beta_3} x_1) \\
& \qquad + \alpha_2 x_1 (1 - F_2(\frac{Q_1}{1-\beta_3} - \frac{1-\beta_1}{1-\beta_3} x_1))] f_1(x_1) dx_1 \\
& + \int_{\frac{Q_1}{1-\beta_1}}^{\infty} [\alpha_3 x_1 (1 - F_2(\frac{x_1(1-\beta_1) - Q_1}{\beta_2})) + \alpha_4 x_1 F_2(\frac{x_1(1-\beta_1) - Q_1}{\beta_2})] f_1(x_1) dx_1 \Big\} \\
& + \Big\{ \int_{-\infty}^{\frac{Q_1}{1-\beta_1}} [\int_{-\infty}^{\frac{Q_1}{1-\beta_3} - \frac{1-\beta_1}{1-\beta_3} x_1} (b_1 - b_2) x_2 f_2(x_2) dx_2 + b_2 \mu_2] f_1(x_1) dx_1 \\
& + \int_{\frac{Q_1}{1-\beta_1}}^{\infty} [\int_{-\infty}^{\frac{x_1(1-\beta_1)-Q_1}{\beta_2}} (b_4 - b_3) x_2 f_2(x_2) dx_2 + b_3 \mu_2] f_1(x_1) dx_1 \Big\} \\
& + Q_1 \Big\{ \int_{-\infty}^{\frac{Q_1}{1-\beta_1}} [\lambda_1 F_2(\frac{Q_1}{1-\beta_3} - \frac{1-\beta_1}{1-\beta_3} x_1) + \lambda_2 (1 - F_2(\frac{Q_1}{1-\beta_3} - \frac{1-\beta_1}{1-\beta_3} x_1))] \\
& + \int_{\frac{Q_1}{1-\beta_1}}^{\infty} [\lambda_3 (1 - F_2(\frac{x_1(1-\beta_1) - Q_1}{\beta_2})) + \lambda_4 F_2(\frac{x_1(1-\beta_1) - Q_1}{\beta_2})] \Big\} \\
= \Big\{ & \alpha_2 \mu_1 + (\alpha_1 - \alpha_2) \int_{-\infty}^{\frac{Q_1}{1-\beta_1}} x_1 f_1(x_1) F_2(\frac{Q_1}{1-\beta_3} - \frac{1-\beta_1}{1-\beta_3} x_1) dx_1 \\
& + (\alpha_4 - \alpha_3) \int_{\frac{Q_1}{1-\beta_1}}^{\infty} x_1 f_1(x_1) F_2(\frac{x_1(1-\beta_1) - Q_1}{\beta_2}) dx_1 \Big\} \\
& + \Big\{ (b_1 - b_2) \int_{-\infty}^{\frac{Q_1}{1-\beta_1}} \int_{-\infty}^{\frac{Q_1}{1-\beta_3} - \frac{1-\beta_1}{1-\beta_3} x_1} x_2 f_2(x_2) dx_2 f_1(x_1) dx_1 \\
& + (b_4 - b_3) \int_{\frac{Q_1}{1-\beta_1}}^{\infty} \int_{-\infty}^{\frac{x_1(1-\beta_1)-Q_1}{\beta_2}} x_2 f_2(x_2) dx_2 f_1(x_1) dx_1 \\
& + b_3 \mu_2 \Big\} Q_1 \Big\{ \lambda_3 + \int_{\frac{Q_1}{1-\beta_1}}^{\infty} (\lambda_4 - \lambda_3) F_2(\frac{x_1(1-\beta_1) - Q_1}{\beta_2}) f_1(x_1) dx_1 \\
& + \int_{-\infty}^{\frac{Q_1}{1-\beta_1}} (\lambda_1 - \lambda_2) F_2(\frac{Q_1}{1-\beta_3} - \frac{1-\beta_1}{1-\beta_3} x_1) f_1(x_1) dx_1 \Big\}
\end{aligned}
\tag{6.50}
$$

For $Q_1 = Q_1^*$, the last term is zero. Thus we can derive the optimal expected profit function:

$$E(\pi(Q_1^*)) = \alpha_2\mu_1 + b_2\mu_2 + (b_1 - b_2)\int_{-\infty}^{\frac{Q_1^*}{1-\beta_1}}\int_{-\infty}^{\frac{Q_1^*}{1-\beta_3}-\frac{1-\beta_1}{1-\beta_3}x_1}(\frac{1-\beta_1}{1-\beta_3}x_1 + x_2)f_2(x_2)dx_2f_1(x_1)dx_1 + (b_4 - b_3)\int_{\frac{Q_1^*}{1-\beta_1}}^{\infty}\int_{-\infty}^{\frac{x_1(1-\beta_1)-Q_1^*}{\beta_2}}(-\frac{1-\beta_1}{\beta_2}x_1 + x_2)f_2(x_2)dx_2f_1(x_1)dx_1 \quad (6.51)$$

Appendix 5: Numerical examples for particular cases

First we consider a special situation: the return probabilities for internet demand by store inventory or drop-shipping option are identical, e.g. $\beta_1 = 0.1$, $\beta_2 = \beta_3 = 0.2$, the expected profit is concave and thus only one solution can be computed using equation 4.17 which is the optimal order quantity: $Q^* = 308$. The first derivative function of the expected profit (equation 6.27) is shown in Figure 6.3.

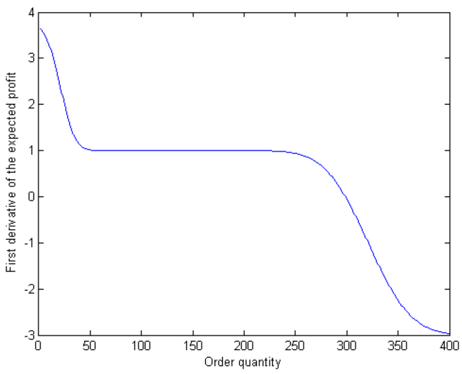


Figure 6.3: First derivative of the expected profit function for identical return probabilities

In the stable stage of the curve, the second term in the right-hand side approximates $-(\lambda_4-\lambda_3)$, the third is approximately zero and fourth equals to zero, thus equation 6.27 has a value of $\lambda_3 = w_2 - w_1 = 1 > 0$, and this positivety is always guaranteed because $w_2 > w_1$ in our model. Thus the optimal order quantity situates after the stable stage as shown in the graph. In the optimal condition, the second term approximates 1, the fourth term equals to zero, we have an approximate equation for the optimal order quantity, which gives the same result $Q^* = 308$:

$$\int_{-\infty}^{\frac{Q_1^*}{1-\beta_1}} F_2(\frac{Q_1^*}{1-\beta_3} - \frac{1-\beta_1}{1-\beta_3}x_1)f_1(x_1)dx_1 \approx \frac{\lambda_3}{\lambda_2-\lambda_1} = \frac{w_2-w_1}{w_2-s} \tag{6.52}$$

In this case, the optimal order quantity is a function of purchasing cost, discount price and return proportions. The return cost doesn't change the value of optimal order quantity.

In normal situations we have similar graph as 6.3 with $\beta_1 < \beta_2$, $\beta_1 < \beta_3$ and $\beta_3 \leqslant \beta_2$.

Abnormal situation is when $\beta_3 > \beta_2$. Three subcases are possible: First case:

$$\beta_2 > \beta_3 - \frac{w_2-w_1}{v_2+w_r-w_1}$$

In this first case, $\lambda_3 > 0$, the stable stage is positive, and the expected profit function is concave. The graph is thus similar to the normal situation.

Second case:

$$\beta_3 - \frac{w_2-s}{v_2+w_r-s} \leqslant \beta_2 \leqslant \beta_3 - \frac{w_2-w_1}{v_2+w_r-w_1}$$

In this second case, the expected profit function is concave. Thus their is only one possible solution for equation 4.17. $\lambda_3 \leqslant 0$, the stable stage is not positive, therefore he optimal order quantity happens before the stable stage. The third term of equation 4.17 approximates zero (Figure 6.4). Thus the optimality condition is:

$$F_{1-\tilde{2}}(\frac{Q_1}{1-\beta_1}) = \frac{\lambda_4}{\lambda_4-\lambda_3}$$

Third case:

$$\beta_2 < \beta_3 - \frac{w_2-s}{v_2+w_r-s}$$

In this third case, the expected profit function is not concave. $\lambda_3 < 0$, the stable stage is negative (Figure 6.4). After the stable stage, the coefficient of the third term in equation 4.17 is positive and thus the first order derivative of expected profit becomes increasing with order quantity. But the first derivative is always negative after the stable stage:

$$\lambda_4 - (\lambda_4-\lambda_3)F_{1-\tilde{2}}(\frac{Q_1}{1-\beta_1}) - (\lambda_2-\lambda_1)F_{1+\tilde{2}}(\frac{Q_1}{1-\beta_1}) \ll$$

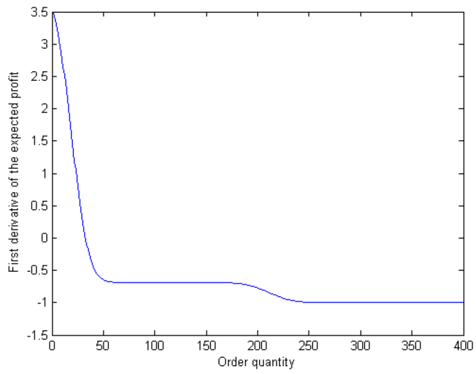


Figure 6.4: First derivative of the expected profit function for identical return probabilities, with $s = 5$, $\beta_1 = 0.1$, $\beta_2 = 0.24$ and $\beta_3 = 0.6$

$$\lambda_4 - (\lambda_4 - \lambda_3) * 1 - (\lambda_2 - \lambda_1) * 1 = \lambda_3 - \lambda_2 + \lambda_1 = \lambda_1 < 0$$

As a result, their is also only one possible solution to equation 4.17. The optimality condition is:

$$F_{1-\tilde{2}}(\frac{Q_1}{1-\beta_1}) = \frac{\lambda_4}{\lambda_4 - \lambda_3}$$

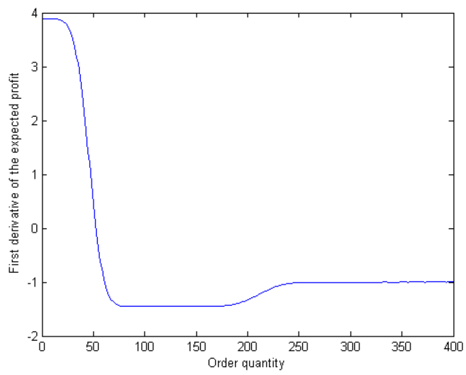


Figure 6.5: First derivative of the expected profit function for identical return probabilities, with $s = 5$, $\beta_1 = 0.1$, $\beta_2 = 0.15$ and $\beta_3 = 0.6$

To conclude from the numerical analysis, for any given value of the variables, only one positive solution for equation 4.17 can exist for normal distributed demand. Thus we can derive the optimal order quantity from equation 4.17 for any cases.

In both the second and third case, the NV has an lower reliance on store inventory. The only necessary condition for it is:

$$\beta_2 \leqslant \beta_3 - \frac{w_2 - w_1}{v_2 + w_r - w_1}$$

The situation for this condition in practice can be that the NV is not good at managing internet sales (e.g. bad packaging), while the drop-shipper is more professional and thus the return proportion of internet business is smaller with drop-shipping option.

Let $\beta_1 = 0.1$, $\beta_2 = 0.2$, $\beta_3 = 0.35$, the optimal order quantity Q^* decreases with w_r (Figure 6.6). This is because $\beta_3 > \beta_2$ and therefore there will be more returns to sell the same amount of products to satisfy internet demand from store inventory than to rely on drop-shipping. When the unit return cost increases, the difference between their return cost is more important, thus the NV stocks less store inventory. The optimal order quantity value falls down at $w_r = 3.75$, this is because $\beta_2 = \beta_3 - \frac{w_2 - w_1}{v_2 + w_r - w_1}$ at $w_r = \frac{11}{3}$ and after this point, the NV relies more on drop-shipping option.

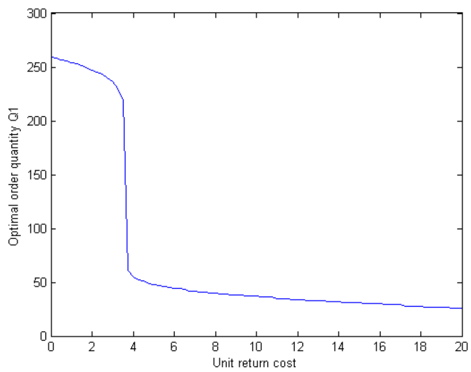


Figure 6.6: Optimal order quantity as a function of unit return cost

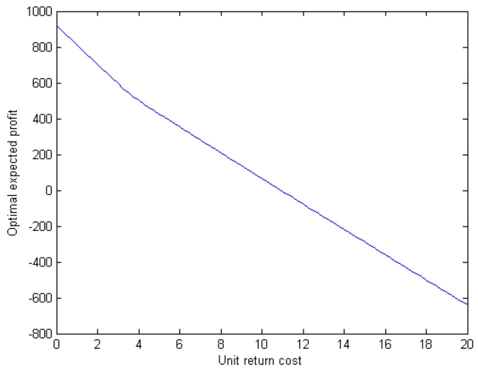


Figure 6.7: Optimal expected profit as a function of unit return cost

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